

# A Generalization of the Cantor-Dedekind Continuum with Nilpotent Infinitesimals

JOSÉ ROQUETTE

*Abstract:* We introduce a generalization of the Cantor-Dedekind continuum with explicit infinitesimals. These infinitesimals are used as numbers obeying the same basic rules as the other elements of the generalized continuum, in accordance with Leibniz's original intuition, but with an important difference: their product is null, as the Dutch theologian Bernard Nieuwentijt sustained, against Leibniz's opinion. The starting-point is the concept of shadow, and from it we define indiscernibility (the central concept) and monad. Monads of points have a global-local nature, because in spite of being infinite-dimensional real affine spaces with the same cardinal as the whole generalized continuum, they are closed intervals with length 0. Monads and shadows (initially defined for points) are then extended to any subset of the new continuum, and their study reveals interesting results of preservation in the areas of set theory and topology. All these concepts do not depend on a definition of limit in the new continuum; yet using them we obtain the basic results of the differential calculus. Finally, we give two examples illustrating how the global-local nature of the monad of a real number can be applied to the differential treatment of certain singularities.

2000 *Mathematics Subject Classification* [\(primary\)](#)

*Keywords:* Infinitesimal methods, indiscernibility, differential calculus, topology, set theory

## 1 Introduction

Up to 1960, when Abraham Robinson created *Non-standard Analysis*, *actual infinitesimals*, i.e. infinitesimals considered as numbers, in the Leibniz's tradition [6], were banished from mathematical analysis by Weierstrass'  $\varepsilon - \delta$  definition of *limit* (in the 1850s), except for a minority of mathematicians and at least one great philosopher (Charles S. Peirce). But physicists and engineers (and differential geometers such as Sophus Lie, Élie Cartan, and Hermann Weyl) refused to deprive themselves of the immense heuristic power of that notion (and rightly so!).

Today, there are two main rigorous theories of actual infinitesimals: *Non-standard Analysis (NSA)* [4], [5], [8], [9], [10], [11], using nonexplicit invertible infinitesimals, and *Smooth Infinitesimal Analysis (SIA)* (F.W. Lawvere, in the late 1960s) [1], [2], [7], with nilpotent infinitesimals (i.e. infinitesimals  $\varepsilon$  such that  $\varepsilon^n = 0$ , for some positive integer  $n$ ). But both theories are considered with suspicion by the immense majority of the mathematical community, and physicists and engineers prefer their strong intuitions.

The generalization ( $\widehat{\mathbb{R}}$ ) of the *usual Cantor-Dedekind continuum* ( $\mathbb{R}$ ) we propose, and the ensuing Calculus, have the following features:

**I** – The elements of  $\widehat{\mathbb{R}}$ , which we call *generalized real numbers*, are the convergent (in the usual sense) sequences in  $\mathbb{R}$ , and those sequences that converge to 0 are called *infinitesimals* (so *infinitesimals are explicit*). The *shadow* of a generalized real number is just its *limit* as a convergent sequence in  $\mathbb{R}$ , and from this concept we define a binary relation on  $\widehat{\mathbb{R}}$  that coincides with the identity of the shadows, and which we call *indiscernibility* ( $\approx$ ). The *monad* of a generalized real number  $x_0$  ( $m_{\approx}(x_0)$ ) is the set of all elements of  $\widehat{\mathbb{R}}$  that are *indiscernible from*  $x_0$ . On the set  $\widehat{\mathbb{R}}$  we define *addition* term by term, but *multiplication* and *ordering* are introduced in a different manner, using the concept of shadow. We obtain an ordered ring extension of  $\mathbb{R}$  (though it is important to take into account **f<sub>2</sub>** below); moreover, the quotient of  $\widehat{\mathbb{R}}$  by  $\approx$  is an ordered field isomorphic to  $\mathbb{R}$ .

Although we can embed  $\mathbb{R}$  in  $\widehat{\mathbb{R}}$  (through the mapping  $\xi \mapsto (\xi)$ , where  $(\xi)$  is the constant sequence determined by the real number  $\xi$ ), we must emphasize two features of  $\widehat{\mathbb{R}}$  that are absent from  $\mathbb{R}$ :

**f<sub>1</sub>**) *The product of two nonnull generalized real numbers or the square of a nonnull generalized real number may be null* (if and only all the factors are *infinitesimal*).

**f<sub>2</sub>**) *Strict ordering is defined on  $\widehat{\mathbb{R}}$  except inside the monads* (as it should be expected, since the elements of the monad of a generalized real number are indiscernible). So we have this version of the usual *trichotomy property*:

$$\left( \forall x, y \in \widehat{\mathbb{R}} \right) (x < y \vee x \approx y \vee y < x) .$$

**II** – We work in two *modes*:

The mode of potentiality, i.e. the totality of notions and concepts that can be defined within the structure  $\mathbb{R}$ .

The mode of actuality, i.e. the totality of notions and concepts that can be defined within the structure  $\widehat{\mathbb{R}}$ , with the exception of any definition of *limit*.

We use the *mode of potentiality* emphasizing the usual definition of *limit*, but in the *mode of actuality*, in the absence of such a definition, we must introduce the fundamental concepts of *generalized real number*, and *shadow*, in the *mode of potentiality*. Nevertheless, we must stress that this translation is only made for the sake of definition: once defined, the two fundamental concepts are used in the *mode of actuality*. Every notion or concept in *the mode of actuality* could be translated into *the mode of potentiality*, but then we would renounce the intuitive and computational power of *actual methods*.

Our work in these two *modes*, sometimes simultaneously (as in the definition of *differentiability*), reflects our conviction that a concept of *actual infinitesimal* and a definition of *limit* are both necessary to a Calculus fit, not only for mathematicians, but also for experimental scientists.

**III** – Each generalized real number  $x$  is indiscernible from exactly one real number: its *shadow*, which we denote by  $\sigma x$ . In fact, *each generalized real number  $x$  admits a unique decomposition as the sum of a real number (its shadow) and an infinitesimal*. We denote this infinitesimal by  $dx$ , and we call it the *differential* of  $x$ . So we have, for each  $x \in \widehat{\mathbb{R}}$ , the unique decomposition, which we call the  $\sigma + d$  decomposition:

$$x = \sigma x + dx.$$

For each  $x \in \widehat{\mathbb{R}}$ , and  $\xi \in \mathbb{R}$ , we have, as a direct consequence of the  $\sigma + d$  decomposition (and we stress its uniqueness!):

$$\sigma \xi = \xi,$$

$$d\xi = 0,$$

$$\sigma(dx) = 0,$$

$$d(dx) = dx,$$

$$\sigma(\xi + dx) = \xi,$$

$$d(\xi + dx) = dx.$$

Although we do not use a definition of *limit* in  $\widehat{\mathbb{R}}$ , we can easily derive the *basic algebraic rules of differentiation*, using the  $\sigma + d$  decomposition.

**IV** – For each subset  $\widehat{A}$  of  $\widehat{\mathbb{R}}$ , we define its *monad* ( $m_{\approx}(\widehat{A})$ ) and *shadow* ( $\sigma(\widehat{A})$ ), and we obtain interesting *set-theoretic* and *topological* results of preservation.

The *intervals* in  $\widehat{\mathbb{R}}$  are simply the monads of the corresponding intervals in  $\mathbb{R}$ , and the *length* of those that are *bounded* (i.e. those intervals in  $\widehat{\mathbb{R}}$  that are monads of bounded intervals in  $\mathbb{R}$ ) is the same as the length of their originals in  $\mathbb{R}$ ; for instance, the *bounded open* and the *bounded closed intervals* in  $\widehat{\mathbb{R}}$  are

$$\begin{aligned}\widehat{]\alpha, \beta[} &:= m_{\approx}(]\alpha, \beta[), \\ \widehat{[\alpha, \beta]} &:= m_{\approx}([\alpha, \beta]),\end{aligned}$$

respectively, where  $\alpha, \beta \in \mathbb{R}$ , and  $\alpha \leq \beta$  (their *length* is  $\beta - \alpha$ ).

Intervals in  $\widehat{\mathbb{R}}$  do not have *pointlike* extremities, and this feature is reminiscent of *Stoic philosophical view* about *segments of Space* or *Time* [12]; for instance, if  $\alpha, \beta, \gamma \in \mathbb{R}$ , and  $\alpha \leq \beta \leq \gamma$ , then

$$\begin{aligned}m_{\approx}(\alpha) &= \widehat{[\alpha, \alpha]} \\ m_{\approx}(\alpha), m_{\approx}(\beta) &\subseteq \widehat{[\alpha, \beta]}, \\ \widehat{[\alpha, \beta]} \cap \widehat{[\beta, \gamma]} &= m_{\approx}(\beta).\end{aligned}$$

**V** – The monad of each generalized real number  $x$  has a *global-local* nature since it is an infinite-dimensional real affine space with the same cardinal as  $\widehat{\mathbb{R}}$  (more precisely,  $|m_{\approx}(x)| = |\widehat{\mathbb{R}}| = 2^{\aleph_0}$ ), yet it is also a closed interval of length 0 (it is easy to prove that  $m_{\approx}(x) = m_{\approx}(\sigma x)$ , so  $m_{\approx}(x) = \widehat{[\sigma x, \sigma x]}$ ).

We use this *dual* nature in two examples of differential treatment of *singularities*.

**VI** – For each function  $\phi : I \rightarrow \mathbb{R}$ , where  $I$  is an open interval in  $\mathbb{R}$ , its *indiscernible extensions* are the functions  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  such that

$$\begin{aligned}f(\sigma x) &= \phi(\sigma x), \\ f(x) &\approx \phi(\sigma x).\end{aligned}$$

If  $\xi_0 \in I$ , and  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  is an indiscernible extension of  $\phi$ , then  $f$  is said to be *differentiable* at  $\xi_0$  iff there exists a real number  $\alpha$  such that

$$(\forall x \in m_{\approx}(\xi_0)) f(x) = \phi(\xi_0) + \alpha dx,$$

with the proviso that  $\alpha := \lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ , when such limit exists in  $\mathbb{R}$ .

$\alpha$  (which is *unique*) is said to be the *derivative* of  $f$  at  $x$ , for each  $x \in m_{\approx}(\xi_0)$ , and we denote it by  $f'(x)$ , as usual.

So we have, when  $f$  is differentiable at  $\xi_0$ :

**d<sub>1</sub>**) If  $x \in m_{\approx}(\xi_0)$ , then  $f'(x) = f'(\xi_0)$ .

**d<sub>2</sub>**) For each  $x \in m_{\approx}(\xi_0)$ ,

$$f(x) = f(\xi_0) + f'(\xi_0)dx.$$

This is the expression, in analytical terms, of the *geometric idea* associated with the concept of *differentiability*, according to Leibniz primeval conception:

*If  $f$  is differentiable at  $\xi_0$ , then the graph of  $f$  coincides locally (i.e. for infinitesimal increments of the argument around  $\xi_0$ ) with its tangent at the point  $(\xi_0, f(\xi_0))$ .*

Notice that if  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  exists in  $\mathbb{R}$  (i.e.  $\phi$  is differentiable at  $\xi_0$ , in the usual sense) and  $f$  is differentiable at  $\xi_0$ , then  $f'(\xi_0)$  is identical with this limit; however,  $f'(\xi_0)$  may exist in the absence of  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ , as it is the case for  $\xi_0 := 0$ ,

$$\text{and } \phi : \mathbb{R} \rightarrow \mathbb{R}, f : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}} \text{ defined by } \phi(\xi) := |\xi|, f(x) := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x \in m_{\approx}(0) \\ -x, & \text{if } x < 0 \end{cases}$$

(clearly,  $f'(0) = 0$ ).

Keeping in mind that the *derivatives* are always associated with *indiscernible extensions*, and using the definition, we obtain not only the *algebraic rules of derivation*, but also fundamental theorems like the **Chain Rule**, the **Inverse Function Theorem**, the **Mean Value Theorem**, and **Taylor's Theorem**.

If  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  exists, for each  $\xi_0 \in I$ , then, among the infinity of indiscernible extensions of  $\phi$ , there exists exactly one that is differentiable at each  $\xi_0 \in I$ ; we call this function the *natural indiscernible extension* of  $\phi$ , and we denote it by  $\hat{\phi}$ .

So  $\hat{\phi} : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  is the function defined by

$$\hat{\phi}(x) := \phi(\xi_0) + \lambda_{\phi}(\xi_0)dx,$$

where  $\lambda_{\phi}(\xi_0)$  denotes  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ .

The concept of *natural indiscernible extension* provides a rule for the definition of the *analogues* (and *extensions*) of the usual functions of Real Analysis. For instance, the natural indiscernible extensions of  $\exp$ ,  $\log$ ,  $\sin$ ,  $\cos$ , are the functions (where  $\widehat{\mathbb{R}}^+$  is the set of positive generalized real numbers):

$$\widehat{\exp} : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}},$$

$$\begin{aligned}\widehat{\log} : \widehat{\mathbb{R}}^+ &\rightarrow \widehat{\mathbb{R}}, \\ \widehat{\sin} : \widehat{\mathbb{R}} &\rightarrow \widehat{\mathbb{R}}, \\ \widehat{\cos} : \widehat{\mathbb{R}} &\rightarrow \widehat{\mathbb{R}},\end{aligned}$$

defined by

$$\begin{aligned}\widehat{\exp}(x) &:= \exp(\sigma x) + \exp(\sigma x)dx, \\ \widehat{\log}(x) &:= \log(\sigma x) + \frac{1}{\sigma x}dx, \\ \widehat{\sin}(x) &:= \sin(\sigma x) + \cos(\sigma x)dx, \\ \widehat{\cos}(x) &:= \cos(\sigma x) - \sin(\sigma x)dx.\end{aligned}$$

We show that these functions have the same basic properties as the usual ones, and we obtain, *rigorously*, some identities that physicists and engineers often use *intuitively*. For example (since  $\sigma(dx) = 0$ , and  $\sigma(1 + dx) = 1$ , as seen in **III**) :

$$\begin{aligned}\widehat{\exp}(dx) &= \exp(\sigma(dx)) + \exp(\sigma(dx))dx = \exp(0) + \exp(0)dx = 1 + dx, \\ \widehat{\log}(1 + dx) &= \log(\sigma(1 + dx)) + \frac{1}{\sigma(1 + dx)}dx = \log(1) + dx = dx, \\ \widehat{\sin}(dx) &= \sin(\sigma(dx)) + \cos(\sigma(dx))dx = \sin(0) + \cos(0)dx = dx, \\ \widehat{\cos}(dx) &= \cos(\sigma(dx)) - \sin(\sigma(dx))dx = \cos(0) - \sin(0)dx = 1.\end{aligned}$$

## 2 The Generalized Real Numbers

Let  $(\mathbb{R}, <, +, \cdot, 0, 1)$  be a model of the usual real number system axioms (in any of the equivalent formulations of most calculus textbooks), and let  $\widehat{\mathbb{R}}$  be the set of all sequences  $x = (\xi_n)$  in  $\mathbb{R}$  that are convergent for the usual absolute value in  $(\mathbb{R}, <, +, \cdot, 0, 1)$ . We refer to  $(\mathbb{R}, <, +, \cdot, 0, 1)$  as the *Cantor-Dedekind continuum*.

**Definition 2.1** Let  $x, y \in \widehat{\mathbb{R}}$ .

If  $\lim x$  is the usual limit of  $x$  in  $(\mathbb{R}, <, +, \cdot, 0, 1)$ , then we call the constant sequence  $(\lim x)$ , the *shadow* of  $x$ , and we denote it by  $\sigma x$ .

$x$  is said to be *indiscernible from*  $y$ , and we denote it by  $x \approx y$ , iff  $x$  and  $y$  have the

same shadow.

$x$  is said to be an *infinitesimal* iff  $x$  is indiscernible from the constant sequence  $(0)$ .

The *monad* of  $x$ , denoted by  $m_{\approx}(x)$ , is the set of all  $y \in \widehat{\mathbb{R}}$  such that  $y$  is indiscernible from  $x$ .

So  $m_{\approx}((0))$  is the set of all infinitesimals.

Clearly, the *indiscernibility* relation,  $\approx$ , is an equivalence relation on  $\widehat{\mathbb{R}}$ , and if  $x$  is an element of  $\widehat{\mathbb{R}}$ , then its equivalence class for  $\approx$  is  $m_{\approx}(x)$ . *Indiscernibility* is the first and more important binary relation defined on  $\widehat{\mathbb{R}}$ .

The next definition introduces a ring structure for  $\widehat{\mathbb{R}}$  with a kind of linear ordering.

**Definition 2.2** On the set  $\widehat{\mathbb{R}}$ , we consider two binary operations, denoted by  $\hat{+}$  and  $\hat{\cdot}$ , and called *addition* and *multiplication*, respectively. If  $x = (\xi_n)$  and  $y = (\eta_n)$  are elements of  $\widehat{\mathbb{R}}$ , then these operations are defined by

$$x \hat{+} y := (\xi_n + \eta_n),$$

$$x \hat{\cdot} y := (\lim x \cdot \eta_n + \lim y \cdot \xi_n - \lim x \cdot \lim y),$$

where at the right-hand of the previous identities we consider the obvious operations on  $\mathbb{R}$  (clearly,  $x \hat{+} y, x \hat{\cdot} y \in \widehat{\mathbb{R}}$  and  $\lim(x \hat{+} y) = \lim x + \lim y$ ,  $\lim(x \hat{\cdot} y) = \lim x \cdot \lim y$ ). We say that  $x$  is *less than*  $y$ , and we denote it by  $x \hat{<} y$ , iff  $\lim x < \lim y$ , and reciprocally, we say that  $x$  is *greater than*  $y$ , and we denote it by  $x \hat{>} y$ , iff  $y \hat{<} x$ , where in  $\lim x < \lim y$  we consider the usual linear ordering on  $\mathbb{R}$ .

The elements of  $\widehat{\mathbb{R}}^+ := \{x \in \widehat{\mathbb{R}} | x \hat{>} (0)\}$  and  $\widehat{\mathbb{R}}^- := \{x \in \widehat{\mathbb{R}} | x \hat{<} (0)\}$  will be called *positive* and *negative*, respectively.

**Proposition 2.3 a)**  $(\widehat{\mathbb{R}}, \hat{<}, \hat{+}, \hat{\cdot}, (0), (1))$  is a commutative ring with the constant sequences  $(0)$  and  $(1)$  as *zero element* and *identity element*, respectively.

**b)** The *shadow mapping*  $\sigma : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$ , defined by  $\sigma(x) := \sigma x$ , is an idempotent ring endomorphism, i.e.

$$(\forall x \in \widehat{\mathbb{R}}) \sigma(\sigma x) = \sigma x;$$

$$(\forall x, y \in \widehat{\mathbb{R}}) \sigma(x \hat{+} y) = \sigma x \hat{+} \sigma y,$$

$$(\forall x, y \in \widehat{\mathbb{R}}) \sigma(x \hat{\cdot} y) = \sigma x \hat{\cdot} \sigma y,$$

$$\sigma(1) = (1).$$

Furthermore,

$$\text{Ker}(\sigma) := \left\{ x \in \widehat{\mathbb{R}} \mid \sigma x = (0) \right\} = m_{\approx}((0)),$$

$$\sigma(\widehat{\mathbb{R}}) = \left\{ y \in \widehat{\mathbb{R}} \mid y \text{ is a constant sequence} \right\}.$$

**c)**  $m_{\approx}((0))$  is a nonnull ideal, so the sum of infinitesimals is an infinitesimal, the additive inverse of an infinitesimal is also an infinitesimal,  $(0)$  is an infinitesimal, the product of an element of  $\widehat{\mathbb{R}}$  and an infinitesimal is still an infinitesimal, and there is a nonnull infinitesimal.

**d)** The product of infinitesimals is always null, i.e.

$$(\forall x, y \in m_{\approx}((0))) x \hat{\cdot} y = (0).$$

In particular, each infinitesimal is nilpotent, since  $x \hat{\cdot} x = (0)$ , for each  $x \in m_{\approx}((0))$ .

**e)** An element of  $\widehat{\mathbb{R}}$  has a *multiplicative inverse* iff it is not an infinitesimal.

**f)** If  $x, y, z \in \widehat{\mathbb{R}}$ , then

$$\neg(x \hat{<} x),$$

$$x \hat{<} y \wedge y \hat{<} z \Rightarrow x \hat{<} z,$$

$$x \hat{<} y \vee x \approx y \vee y \hat{<} x,$$

$$x \hat{<} y \Rightarrow x \hat{+} z \hat{<} y \hat{+} z,$$

$$x \hat{<} y \wedge z \hat{>} (0) \Rightarrow x \hat{\cdot} z \hat{<} y \hat{\cdot} z.$$

So, if we adopt the version of the usual *trichotomy property* expressed by the third formula above, then  $(\widehat{\mathbb{R}}, \hat{<}, \hat{+}, \hat{\cdot}, (0), (1))$  may be considered an ordered ring.

**g)**  $(\widehat{\mathbb{R}}, \hat{<}, \hat{+}, \hat{\cdot}, (0), (1))$  is *archimedean*, i.e.

$$(\forall x, y \in \widehat{\mathbb{R}})(x \hat{>} (0) \Rightarrow (\exists m \in \mathbb{N}) \widehat{m}x \hat{>} y),$$

where  $\widehat{m}x$  abbreviates  $x_1 \hat{+} x_2 \hat{+} \dots \hat{+} x_m$ , when  $x_1 = x_2 = \dots = x_m = x$  (assuming  $\widehat{1}x = x$ ).

**h)** The mapping  $*$  :  $\mathbb{R} \rightarrow \sigma(\widehat{\mathbb{R}})$ , defined by  $*(\xi) := (\xi)$ , where  $(\xi)$  is the usual constant sequence determined by  $\xi$ , is a ring isomorphism of  $(\mathbb{R}, <, +, \cdot, 0, 1)$  onto  $(\sigma(\widehat{\mathbb{R}}), \hat{<}, \hat{+}, \hat{\cdot}, (0), (1))$ , and

$$(\forall \xi, \eta \in \mathbb{R})(\xi < \eta \Leftrightarrow *(\xi) \hat{<} *(\eta)).$$

So, using  $*$ , we can embed  $(\mathbb{R}, <, +, \cdot, 0, 1)$  in  $(\widehat{\mathbb{R}}, \hat{<}, \hat{+}, \hat{\cdot}, (0), (1))$ .

**Proof a)** Only the proofs of the associative property of multiplication and the distributive property of multiplication over addition offer some (slight) difficulty.



If  $x = (\xi_n), y = (\eta_n), z = (\zeta_n) \in \widehat{\mathbb{R}}$ , then

$$\begin{aligned} (x \hat{\cdot} y) \hat{\cdot} z &= (\lim x \cdot \eta_n + \lim y \cdot \xi_n - \lim x \cdot \lim y) \hat{\cdot} z = (\lim x \cdot \lim y \cdot \zeta_n + \\ &+ \lim z \cdot \lim x \cdot \eta_n + \lim z \cdot \lim y \cdot \xi_n - \lim z \cdot \lim x \cdot \lim y - \lim x \cdot \lim y \cdot \lim z) = \\ &= (\lim x \cdot \lim y \cdot \zeta_n + \lim z \cdot \lim x \cdot \eta_n + \lim z \cdot \lim y \cdot \xi_n - 2 \lim x \cdot \lim y \cdot \lim z), \\ x \hat{\cdot} (y \hat{\cdot} z) &= x \hat{\cdot} (\lim y \cdot \zeta_n + \lim z \cdot \eta_n - \lim y \cdot \lim z) = (\lim x \cdot \lim y \cdot \zeta_n + \\ &+ \lim x \cdot \lim z \cdot \eta_n - \lim x \cdot \lim y \cdot \lim z + \lim y \cdot \lim z \cdot \xi_n - \lim x \cdot \lim y \cdot \lim z) = \\ &= (\lim x \cdot \lim y \cdot \zeta_n + \lim x \cdot \lim z \cdot \eta_n + \lim y \cdot \lim z \cdot \xi_n - 2 \lim x \cdot \lim y \cdot \lim z) = \\ &= (x \hat{\cdot} y) \hat{\cdot} z; \end{aligned}$$

$$\begin{aligned} x \hat{\cdot} (y \hat{+} z) &= x \hat{\cdot} (\eta_n + \zeta_n) = (\lim x \cdot \eta_n + \lim x \cdot \zeta_n + \lim y \cdot \xi_n + \\ &+ \lim z \cdot \xi_n - \lim x \cdot \lim y - \lim x \cdot \lim z) = (\lim x \cdot \eta_n + \lim y \cdot \xi_n - \\ &- \lim x \cdot \lim y) \hat{+} (\lim x \cdot \zeta_n + \lim z \cdot \xi_n - \lim x \cdot \lim z) = (x \hat{\cdot} y) \hat{+} (x \hat{\cdot} z). \end{aligned}$$

**b)** is an immediate consequence of the usual algebraic properties of limits, and **c)**, **d)** follow easily from **a)**, **b)**.

**e)** If  $x = (\xi_n) \in \widehat{\mathbb{R}}$  and  $x$  is not infinitesimal, then a direct calculation shows that

$$x \hat{\cdot} \left( \frac{1}{\lim x} - \frac{\xi_n - \lim x}{(\lim x)^2} \right) = (1);$$

so, since multiplication on  $\widehat{\mathbb{R}}$  is associative, commutative, and (1) is its identity element,  $\left( \frac{1}{\lim x} - \frac{\xi_n - \lim x}{(\lim x)^2} \right)$  is the *multiplicative inverse* of  $x = (\xi_n)$ .

If  $x$  is infinitesimal, then we have (see **a)** and **b)**), for each  $y \in \widehat{\mathbb{R}}$ :

$$\sigma(x \hat{\cdot} y) = \sigma x \hat{\cdot} \sigma y = (0) \hat{\cdot} \sigma y = (0) \neq (1),$$

and we conclude that  $x$  is not invertible.

Finally, **f)**, **g)**, **h)** admit a quite straightforward proof. ■

**Remark 2.4** In accordance with **proposition 2.3 h)**, we identify  $\mathbb{R}$  with  $\sigma(\widehat{\mathbb{R}})$  and  $\xi$  with  $(\xi)$ , for each  $\xi \in \mathbb{R}$ . For instance, we identify 0 with the infinite sequence (0) and, for each  $x \in \widehat{\mathbb{R}}$ ,  $\xi \in \mathbb{R}$ , we identify  $\lim x$  with  $\sigma x$  and  $\xi$  with  $\sigma \xi$ . Furthermore, from now on we shall use the symbols  $+$ ,  $\cdot$ ,  $<$  not only for the usual addition, multiplication and linear ordering on  $\mathbb{R}$ , but also for the corresponding binary operations and relation  $\hat{+}$ ,  $\hat{\cdot}$ ,  $\hat{<}$  on  $\widehat{\mathbb{R}}$ , and we shall even drop the symbol  $\cdot$  in most formulas. For example, revisiting part of **definition 2.2**, we have, for each  $x, y \in \widehat{\mathbb{R}}$ :

$$x < y \Leftrightarrow \sigma x < \sigma y.$$

For the additive and multiplicative powers, we simply write  $mx$  and  $x^m$  instead of  $\widehat{m}x$  and  $x^{\widehat{m}}$  (where  $x^{\widehat{m}}$  abbreviates  $x_1 \hat{\cdot} x_2 \hat{\cdot} \dots \hat{\cdot} x_m$ , when  $x_1 = x_2 = \dots = x_m = x$  (assuming  $x^{\widehat{1}} = x$ )), respectively.

In the spirit of these identifications and notational simplifications, notice that if  $\xi \in \mathbb{R}$  and  $x \in \widehat{\mathbb{R}}$ , then  $\xi x$  (previously denoted by  $(\xi) \hat{\cdot} x$ ) coincides with the result of the scalar multiplication of the real number  $\xi$  by the sequence  $x$ .

If  $x, y \in \widehat{\mathbb{R}}$  and  $x$  is not an infinitesimal, then we denote the *multiplicative inverse* of  $x$  by  $x^{-1}$  or  $\frac{1}{x}$ ; so  $\frac{1}{x} = \left( \frac{1}{\lim x} - \frac{\xi_n - \lim x}{(\lim x)^2} \right)$ . We also denote  $yx^{-1}$  (*the quotient* of  $y$  by  $x$ ) by  $\frac{y}{x}$ , as usual.

We maintain the general designation of *real numbers* for the elements of  $\mathbb{R}$  and call the elements of  $\widehat{\mathbb{R}}$  *generalized real numbers*.

Let us see some explicit *generalized real numbers* (by *explicit* we mean *unambiguously defined as a convergent sequence of real numbers*):

**Example 2.5 1)** The eventually null sequences  $(1, 0, 0, 0, \dots)$ ,  $(0, 1, 0, 0, 0, \dots)$ ,  $(0, 0, 1, 0, 0, 0, \dots)$ ,  $\dots$  are nonnull infinitesimal elements of  $\widehat{\mathbb{R}}$ . So we can exhibit *nonnull infinitesimals*.

**2)** Let  $\xi_0$  be a nonnull real number. Then:

The sequences  $(0, \xi_0, \xi_0, \xi_0, \dots)$ ,  $(0, 0, \xi_0, \xi_0, \xi_0, \dots)$ ,  $(0, 0, 0, \xi_0, \xi_0, \xi_0, \dots)$ ,  $\dots$  are different elements of  $m_{\approx}(\xi_0) \setminus \{\xi_0\}$ .

In the next proposition, which admits a simple proof, **e)** and **f)** are particularly important.

**Proposition 2.6 a)**  $\left( \forall x \in \widehat{\mathbb{R}} \right) (x = \sigma x \Leftrightarrow x \in \mathbb{R})$ .

**b)**  $(\forall \xi, \eta \in \mathbb{R}) (\xi \approx \eta \Leftrightarrow \xi = \eta)$ .

**c)**  $\mathbb{R} \cap m_{\approx}(0) = \{0\}$ .

**d)** Infinitesimals are not comparable with respect to the binary relation  $<$  on  $\widehat{\mathbb{R}}$ , i.e. if  $\hat{\varepsilon}$  and  $\hat{\delta}$  are infinitesimals, then

$$\neg(\hat{\varepsilon} < \hat{\delta}) \wedge \neg(\hat{\delta} < \hat{\varepsilon}),$$

**e)** An infinitesimal is less than any positive generalized real number and greater than any negative generalized real number, i.e. if  $\hat{\varepsilon}$  is an infinitesimal, then

$$(\forall x \in \widehat{\mathbb{R}}^+) \hat{\varepsilon} < x,$$

$$(\forall y \in \widehat{\mathbb{R}}^-) \hat{\varepsilon} > y.$$

In particular:

$$(\forall \xi \in \mathbb{R}^+) \hat{\varepsilon} < \xi,$$

$$(\forall \eta \in \mathbb{R}^-) \hat{\varepsilon} > \eta,$$

where  $\mathbb{R}^+$  and  $\mathbb{R}^-$  are the usual sets of (strictly) positive and (strictly) negative real numbers, respectively (notice that  $\mathbb{R}^+ \subseteq \hat{\mathbb{R}}^+$  and  $\mathbb{R}^- \subseteq \hat{\mathbb{R}}^-$ , by **proposition 2.3 h**).

**f)** Each generalized real number is indiscernible from exactly one real number: its shadow, i.e.

$$(\forall x \in \hat{\mathbb{R}})(x \approx \sigma x \wedge (\forall \xi \in \mathbb{R})(x \approx \xi \Rightarrow \xi = \sigma x)).$$

### 3 The $\sigma + d$ Decomposition

As a direct consequence of **proposition 2.3 a), b)**, we have:

**Proposition 3.1** If  $x$  is a generalized real number, then there is a unique infinitesimal  $\hat{\varepsilon}(x)$  such that

$$x = \sigma x + \hat{\varepsilon}(x).$$

**Definition 3.2** If  $x$  is a generalized real number, then we denote  $\hat{\varepsilon}(x)$  by  $dx$ , and we call it the *differential* of  $x$ .

**Proposition 3.3** If  $x$  is a generalized real number then  $x = \sigma x + dx$  is the unique decomposition of  $x$  as the sum of a real number and an infinitesimal.

**Proof.** We just have to use **proposition 2.3 a), c)**, **proposition 2.6 c)**, **proposition 3.1**, and, of course, **definition 3.2**. ■

We call the decomposition stated by the previous proposition, the  $\sigma + d$  decomposition. Notice that the *differential* of a generalized real number  $x$  is already inlaid in  $x$ , and since  $\sigma x$  and  $dx$  are a constant sequence and a sequence converging to 0, in  $\mathbb{R}$ , we are entitled to express the following intuition: a generalized real number has a unique decomposition as the sum of a *static part* (its *shadow*) and a *dynamic part* (its *differential*).

Clearly:

**Corollary 3.4 a)**  $(\forall x \in \widehat{\mathbb{R}})(dx = 0 \Leftrightarrow x \in \mathbb{R})$ .

**b)**  $(\forall x \in \widehat{\mathbb{R}})(x = dx \Leftrightarrow x \approx 0)$ .

**c)**  $(\forall x \in \widehat{\mathbb{R}})d(dx) = dx$ .

The following lemma is the key to obtain the basic algebraic rules of differentiation.

**Lemma 3.5 a)** If  $x, y \in \widehat{\mathbb{R}}$ , then

$$x + y = \sigma x + \sigma y + dx + dy,$$

$$x - y = \sigma x - \sigma y + dx - dy.$$

**b)** If  $x, y \in \widehat{\mathbb{R}}$ , then

$$xy = (\sigma x)(\sigma y) + (\sigma x)dy + (\sigma y)dx = (\sigma x)(\sigma y) + xdy + ydx.$$

In particular, for each  $\xi \in \mathbb{R}$ :

$$\xi x = \xi(\sigma x) + \xi dx.$$

**c)** If  $m \in \mathbb{N}$ , and  $x \in \widehat{\mathbb{R}}$ , then (with  $x^0 = 1$ )

$$x^m = (\sigma x)^m + m(\sigma x)^{m-1}dx = (\sigma x)^m + mx^{m-1}dx.$$

**d)** If  $x \in \widehat{\mathbb{R}}$ , and  $x$  is not an infinitesimal, then

$$\frac{1}{x} = \frac{1}{\sigma x} - \frac{1}{(\sigma x)^2}dx = \frac{1}{\sigma x} - \frac{1}{x^2}dx.$$

**e)** If  $x, y \in \widehat{\mathbb{R}}$ , and  $x$  is not an infinitesimal, then

$$\frac{y}{x} = \frac{\sigma y}{\sigma x} + \frac{(\sigma x)dy - (\sigma y)dx}{(\sigma x)^2} = \frac{\sigma y}{\sigma x} + \frac{xdy - ydx}{x^2}.$$

**f)** If  $x \in \widehat{\mathbb{R}}^+$ ,  $m \in \mathbb{N}$  and  $m > 1$ , then there is a unique  $y \in \widehat{\mathbb{R}}^+$  such that

$$y^m = x.$$

Such  $y$  will be denoted by  $\sqrt[m]{x}$ , and we have:

$$\sqrt[m]{x} = \sqrt[m]{\sigma x} + \frac{1}{m \sqrt[m]{(\sigma x)^{m-1}}}dx = \sqrt[m]{\sigma x} + \frac{1}{m \sqrt[m]{x^{m-1}}}dx,$$

where  $\sqrt[m]{\sigma x}$  and  $\sqrt[m]{(\sigma x)^{m-1}}$  are the usual *positive  $m$ th roots* of  $\sigma x$  and  $(\sigma x)^{m-1}$ , respectively.

**Proof** Only the proof of **f)** has some difficulty.

If  $x, y \in \widehat{\mathbb{R}}^+$ , then  $\sigma x > 0$  and  $\sigma y > 0$ .

So, using **c)** and **proposition 3.3**, we have:

$$\begin{aligned} y^m = x &\Leftrightarrow (\sigma y + dy)^m = \sigma x + dx \Leftrightarrow (\sigma y)^m + m(\sigma y)^{m-1}dy = \sigma x + dx \Leftrightarrow \\ &\Leftrightarrow \begin{cases} \sigma y = \sqrt[m]{\sigma x} \\ dy = \frac{1}{m \sqrt[m]{(\sigma x)^{m-1}}} dx \end{cases} \Leftrightarrow y = \sqrt[m]{\sigma x} + \frac{1}{m \sqrt[m]{(\sigma x)^{m-1}}} dx. \end{aligned}$$

But  $\sqrt[m]{\sigma x} > 0$ , since  $\sigma x > 0$ ; so

$$\sqrt[m]{\sigma x} + \frac{1}{m \sqrt[m]{(\sigma x)^{m-1}}} dx > 0.$$

We have proven the existence (and uniqueness) of  $\sqrt[m]{x}$  and the identity

$$\sqrt[m]{x} = \sqrt[m]{\sigma x} + \frac{1}{m \sqrt[m]{(\sigma x)^{m-1}}} dx.$$

In particular, if  $x \in \mathbb{R}^+$ , then

$$\sqrt[m]{x} = \sqrt[m]{\sigma x}$$

Using **c)** and the result already proved (notice that  $x^{m-1} > 0$ , since  $\sigma(x^{m-1}) = (\sigma x)^{m-1} > 0$ ), we obtain:

$$\sqrt[m]{x^{m-1}} = \sqrt[m]{(\sigma x)^{m-1}} + \hat{\varepsilon},$$

where  $\hat{\varepsilon}$  is the infinitesimal defined by

$$\hat{\varepsilon} := \frac{1}{m \sqrt[m]{(\sigma x)^{(m-1)^2}} (m-1)(\sigma x)^{m-2}} dx.$$

Then, using **d)**,

$$\frac{1}{\sqrt[m]{x^{m-1}}} = \frac{1}{\sqrt[m]{(\sigma x)^{m-1}}} - \frac{1}{\left(\sqrt[m]{(\sigma x)^{m-1}}\right)^2 \hat{\varepsilon}}.$$

Since the product of infinitesimals is 0, we have:

$$\frac{1}{\sqrt[m]{x^{m-1}}} dx = \frac{1}{\sqrt[m]{(\sigma x)^{m-1}}} dx. \blacksquare$$

As an immediate consequence of the previous lemma, we obtain, using **proposition 3.3**, the basic algebraic rules of differentiation, without using any notion of *limit* in  $\widehat{\mathbb{R}}$ :

**Proposition 3.6 a)** If  $x, y \in \widehat{\mathbb{R}}$ , then

$$d(x + y) = dx + dy,$$

$$d(x - y) = dx - dy.$$

**b)** If  $x, y \in \widehat{\mathbb{R}}$ , then

$$d(xy) = (\sigma x)dy + (\sigma y)dx = xdy + ydx.$$

In particular, for each  $\xi \in \mathbb{R}$ :

$$d(\xi x) = \xi dx.$$

**c)** If  $m \in \mathbb{N}$ , and  $x \in \widehat{\mathbb{R}}$ , then

$$d(x^m) = m(\sigma x)^{m-1} dx = mx^{m-1} dx.$$

**d)** If  $x \in \widehat{\mathbb{R}}$ , and  $x$  is not an infinitesimal, then

$$d\left(\frac{1}{x}\right) = -\frac{1}{(\sigma x)^2} dx = -\frac{1}{x^2} dx.$$

**e)** If  $x, y \in \widehat{\mathbb{R}}$ , and  $x$  is not an infinitesimal, then

$$d\left(\frac{y}{x}\right) = \frac{(\sigma x) dy - (\sigma y) dx}{(\sigma x)^2} = \frac{xdy - ydx}{x^2}.$$

**f)** If  $x \in \widehat{\mathbb{R}}^+$ ,  $m \in \mathbb{N}$  and  $m > 1$ , then

$$d(\sqrt[m]{x}) = \frac{1}{m \sqrt[m]{(\sigma x)^{m-1}}} dx = \frac{1}{m \sqrt[m]{x^{m-1}}} dx.$$

We close this section with a density theorem, and a theorem relating the *generalized real continuum*,  $(\widehat{\mathbb{R}}, <, +, \cdot, 0, 1)$ , to the *Cantor-Dedekind continuum*.

### Theorem 3.7 (The Density Theorem)

**a)** If  $x$  and  $y$  are generalized real numbers such that  $x < y$ , then there exists  $\zeta \in \mathbb{R}$  such that  $x < \zeta < y$ .

**b)** If  $\xi$  and  $\eta$  are real numbers such that  $\xi < \eta$ , then there exists  $z \in \widehat{\mathbb{R}} \setminus \mathbb{R}$  such that  $\xi < z < \eta$ .

**Proof** **a)** We may choose  $\varsigma = \frac{\sigma x + \sigma y}{2}$ .

**b)** If  $\hat{\varepsilon}$  is an infinitesimal and  $\hat{\varepsilon} \neq 0$ , then we may choose  $z = \frac{\xi + \eta}{2} + \hat{\varepsilon}$ . ■

We already mentioned the trivial facts that  $\approx$  is an equivalence relation on  $\widehat{\mathbb{R}}$ , and the equivalence class of each  $x \in \widehat{\mathbb{R}}$  is  $m_{\approx}(x) = x + m_{\approx}(0)$ . On the *quotient* of  $\widehat{\mathbb{R}}$  by  $\approx$ ,

i.e. the set  $\widehat{\mathbb{R}}/\approx := \{m_{\approx}(x) | x \in \widehat{\mathbb{R}}\}$ , we consider now two binary operations, denoted by  $\boxplus$  and  $\boxtimes$ , and called *addition* and *multiplication*, respectively, and a binary relation denoted by  $\sqsubset$ . These operations and relation are defined by:

$$m_{\approx}(x) \boxplus m_{\approx}(y) := m_{\approx}(x + y),$$

$$m_{\approx}(x) \boxtimes m_{\approx}(y) := m_{\approx}(xy),$$

$$m_{\approx}(x) \sqsubset m_{\approx}(y) :\Leftrightarrow x < y,$$

using, at the right-hand of the previous identities, the obvious binary operations and relation on  $\widehat{\mathbb{R}}$ .

It is a simple task to show that  $\boxplus$ ,  $\boxtimes$ ,  $\sqsubset$  are well-defined, and to prove the next theorem.

**Theorem 3.8 a)**  $(\widehat{\mathbb{R}}/\approx, \sqsubset, \boxplus, \boxtimes, m_{\approx}(0), m_{\approx}(1))$  is an ordered field with  $m_{\approx}(0)$  and  $m_{\approx}(1)$  as *zero* and *identity elements*, respectively.

**b)** The mapping  $\phi : \widehat{\mathbb{R}}/\approx \rightarrow \mathbb{R}$ , defined by  $\phi(m_{\approx}(x)) := \sigma x$ , is an ordered field isomorphism of  $(\widehat{\mathbb{R}}/\approx, \sqsubset, \boxplus, \boxtimes, m_{\approx}(0), m_{\approx}(1))$  onto the *Cantor-Dedekind continuum*,  $(\mathbb{R}, <, +, \cdot, 0, 1)$ ; so if we denote these fields simply by  $\widehat{\mathbb{R}}/\approx$  and  $\mathbb{R}$ , we have:

$$\widehat{\mathbb{R}}/\approx \cong \mathbb{R},$$

i.e.  $\widehat{\mathbb{R}}/\approx$  is isomorphic to  $\mathbb{R}$ .

As we have just seen:

*If we take the monads in the structure  $\widehat{\mathbb{R}}$  for points, as we do in the structure  $\widehat{\mathbb{R}}/\approx$ , then we obtain the Cantor-Dedekind continuum. Otherwise, we have a richer continuum with indiscernibility and nilpotent infinitesimals.*

## 4 Monads and Shadows

The next two propositions show that  $\{m_{\approx}(x) | x \in \widehat{\mathbb{R}}\}$  is a partition of  $\widehat{\mathbb{R}}$  into infinite-dimensional real affine spaces, each one with the same cardinal as  $\widehat{\mathbb{R}}$ , and this is also true for  $\{m_{\approx}(\xi) | \xi \in \mathbb{R}\}$  (since  $m_{\approx}(x) = m_{\approx}(\sigma x)$ , for each  $x \in \widehat{\mathbb{R}}$ ).

**Proposition 4.1** The monad of each generalized real number has the same cardinal as  $\widehat{\mathbb{R}}$ .

**Proof** Since  $m_{\approx}(x) = m_{\approx}(\sigma x)$ , for each  $x \in \widehat{\mathbb{R}}$ , we may prove the proposition only for the monads of real numbers.

Let  $\xi \in \mathbb{R}$ , and let  $\widehat{\mathbb{R}}_{\xi}$  be the set of all generalized real numbers  $x = (\xi_n)$  such that  $\xi_n = \xi$ , for  $n > 1$ . Then (denoting by  $|\widehat{A}|$  the cardinal of each subset  $\widehat{A}$  of  $\widehat{\mathbb{R}}$ ):

$$|\widehat{\mathbb{R}}_{\xi}| \leq |m_{\approx}(\xi)| \leq |\mathbb{R}^{\mathbb{N}}|,$$

where  $\mathbb{R}^{\mathbb{N}}$  denotes the set of all sequences in  $\mathbb{R}$ .

Obviously,

$$|\widehat{\mathbb{R}}_{\xi}| = |\mathbb{R}| = 2^{\aleph_0},$$

and

$$|\mathbb{R}^{\mathbb{N}}| = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0}.$$

So

$$|m_{\approx}(\xi)| = 2^{\aleph_0}.$$

Finally,

$$|\widehat{\mathbb{R}}| = |\cup \{m_{\approx}(\xi) | \xi \in \mathbb{R}\}| = 2^{\aleph_0} 2^{\aleph_0} = 2^{\aleph_0}. \blacksquare$$

**Proposition 4.2 a)**  $m_{\approx}(0)$  is an infinite-dimensional real vector space, if we consider addition and multiplication defined on  $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ , as *vector addition* and *scalar multiplication* defined on  $m_{\approx}(0) \times m_{\approx}(0)$  and  $\mathbb{R} \times m_{\approx}(0)$ , respectively. Moreover,  $m_{\approx}(0)$  contains the *real spaces*  $\mathcal{P}^p$ , for each  $p \in [1, +\infty[$ .

**b)** If we consider  $m_{\approx}(0)$  with the structure of real vector space mentioned in **a)**, then

$m_{\approx}(x)$  is an infinite-dimensional real affine space, for each  $x \in \widehat{\mathbb{R}}$ .

**Proof a)** It is trivial to prove that  $m_{\approx}(0)$  is a real vector space, using **proposition 2.3 a), c)**. Finally, if  $p \in [1, +\infty[$  and  $x = (\xi_n) \in \mathcal{P}^p$ , then  $\sum_{n=1}^{+\infty} |\xi_n|^p < +\infty$  and, consequently,  $x = (\xi_n) \in m_{\approx}(0)$ . **b)** follows from **a)**, since  $m_{\approx}(x) = x + m_{\approx}(0)$ , for each  $x \in \widehat{\mathbb{R}}$ .  $\blacksquare$

The next definition generalizes the concepts of *monad* and *shadow* to any subset of  $\widehat{\mathbb{R}}$ .



**Definition 4.3** Let  $\hat{A} \subseteq \hat{\mathbb{R}}$ .

The *monad* of  $\hat{A}$  and the *shadow* of  $\hat{A}$ , denoted by  $m_{\approx}(\hat{A})$  and  $\sigma(\hat{A})$ , respectively, are defined by:

$$m_{\approx}(\hat{A}) := \cup \left\{ m_{\approx}(x) \mid x \in \hat{A} \right\},$$

$$\sigma(\hat{A}) := \cup \left\{ \{\sigma x\} \mid x \in \hat{A} \right\}.$$

So

$$m_{\approx}(\hat{A}) = \left\{ x \in \hat{\mathbb{R}} \mid (\exists y \in \hat{A}) x \approx y \right\},$$

$$\sigma(\hat{A}) = \left\{ \sigma x \mid x \in \hat{A} \right\}.$$

Clearly, we have, for each  $x \in \hat{\mathbb{R}}$  and  $\hat{A} \subseteq \hat{\mathbb{R}}$ ,

$$m_{\approx}(\{x\}) = m_{\approx}(x),$$

$$\sigma(\{x\}) = \{\sigma x\},$$

$$\hat{A} \subseteq m_{\approx}(\hat{A}).$$

The next three propositions state some basic properties of *monads* and *shadows*, and admit quite straightforward proofs.

**Proposition 4.4** Let  $\hat{A}, \hat{B} \subseteq \hat{\mathbb{R}}$ . Then:

- a)  $\hat{A} \subseteq \hat{B} \Rightarrow m_{\approx}(\hat{A}) \subseteq m_{\approx}(\hat{B}) \wedge \sigma(\hat{A}) \subseteq \sigma(\hat{B})$ .
- b)  $\hat{A} \subseteq \mathbb{R} \Leftrightarrow \sigma(\hat{A}) = \hat{A}$ .
- c)  $m_{\approx}(\sigma(\hat{A})) = m_{\approx}(\hat{A}) \wedge \sigma(m_{\approx}(\hat{A})) = \sigma(\hat{A})$ .
- d)  $\hat{A}, \hat{B} \subseteq \mathbb{R} \Rightarrow (m_{\approx}(\hat{A}) = m_{\approx}(\hat{B}) \Leftrightarrow \hat{A} = \hat{B})$ .

The *monad* and *shadow operators* on subsets of  $\hat{\mathbb{R}}$  preserve the Boolean operations on sets, with some looseness in the case of *intersection* and *complement* (this is the core information expressed in the next two propositions).

**Proposition 4.5** a)  $m_{\approx}(\emptyset) = \emptyset$ ,  $m_{\approx}(\widehat{\mathbb{R}}) = m_{\approx}(\mathbb{R}) = \widehat{\mathbb{R}}$ .

Let  $\widehat{A}, \widehat{B} \subseteq \widehat{\mathbb{R}}$ . Then:

- b)  $m_{\approx}(m_{\approx}(\widehat{A})) = m_{\approx}(\widehat{A})$ ,
- c)  $m_{\approx}(\widehat{A} \cup \widehat{B}) = m_{\approx}(\widehat{A}) \cup m_{\approx}(\widehat{B})$ ,
- d)  $m_{\approx}(\widehat{A} \cap \widehat{B}) \subseteq m_{\approx}(\widehat{A}) \cap m_{\approx}(\widehat{B})$ ,
- $m_{\approx}(\widehat{A}) \setminus m_{\approx}(\widehat{B}) \subseteq m_{\approx}(\widehat{A} \setminus \widehat{B})$ .

If  $\widehat{A}, \widehat{B} \subseteq \mathbb{R}$ , then

$$\begin{aligned} m_{\approx}(\widehat{A} \cap \widehat{B}) &= m_{\approx}(\widehat{A}) \cap m_{\approx}(\widehat{B}), \\ m_{\approx}(\widehat{A} \setminus \widehat{B}) &= m_{\approx}(\widehat{A}) \setminus m_{\approx}(\widehat{B}). \end{aligned}$$

Let  $\widehat{\mathbf{A}} \subseteq P(\widehat{\mathbb{R}})$  (i.e.  $\widehat{\mathbf{A}}$  is a collection of subsets of  $\widehat{\mathbb{R}}$ ). Then:

- e)  $m_{\approx}(\cup \{\widehat{A} \mid \widehat{A} \in \widehat{\mathbf{A}}\}) = \cup \{m_{\approx}(\widehat{A}) \mid \widehat{A} \in \widehat{\mathbf{A}}\}$ ,
- f)  $m_{\approx}(\cap \{\widehat{A} \mid \widehat{A} \in \widehat{\mathbf{A}}\}) \subseteq \cap \{m_{\approx}(\widehat{A}) \mid \widehat{A} \in \widehat{\mathbf{A}}\}$ .

If  $\widehat{\mathbf{A}} \subseteq P(\mathbb{R})$  (i.e.  $\widehat{\mathbf{A}}$  is a collection of subsets of  $\mathbb{R}$ ), then

$$m_{\approx}(\cap \{\widehat{A} \mid \widehat{A} \in \widehat{\mathbf{A}}\}) = \cap \{m_{\approx}(\widehat{A}) \mid \widehat{A} \in \widehat{\mathbf{A}}\}.$$

**Proposition 4.6** a)  $\sigma(\emptyset) = \emptyset$ ,  $\sigma(\widehat{\mathbb{R}}) = \sigma(\mathbb{R}) = \mathbb{R}$ .

Let  $\widehat{A}, \widehat{B} \subseteq \widehat{\mathbb{R}}$ . Then:

- b)  $\sigma(\sigma(\widehat{A})) = \sigma(\widehat{A})$ ,
- c)  $\sigma(\widehat{A} \cup \widehat{B}) = \sigma(\widehat{A}) \cup \sigma(\widehat{B})$ ,
- d)  $\sigma(\widehat{A} \cap \widehat{B}) \subseteq \sigma(\widehat{A}) \cap \sigma(\widehat{B})$ ,
- $\sigma(\widehat{A}) \setminus \sigma(\widehat{B}) \subseteq \sigma(\widehat{A} \setminus \widehat{B})$ .

If  $\widehat{A}$  and  $\widehat{B}$  are monads of subsets of  $\mathbb{R}$ , then

$$\begin{aligned} \sigma(\widehat{A} \cap \widehat{B}) &= \sigma(\widehat{A}) \cap \sigma(\widehat{B}), \\ \sigma(\widehat{A} \setminus \widehat{B}) &= \sigma(\widehat{A}) \setminus \sigma(\widehat{B}). \end{aligned}$$

Let  $\hat{\mathbf{A}} \subseteq P(\hat{\mathbb{R}})$  (i.e.  $\hat{\mathbf{A}}$  is a collection of subsets of  $\hat{\mathbb{R}}$ ). Then:

$$\text{e) } \sigma\left(\bigcup \left\{\hat{A} \mid \hat{A} \in \hat{\mathbf{A}}\right\}\right) = \bigcup \left\{\sigma(\hat{A}) \mid \hat{A} \in \hat{\mathbf{A}}\right\},$$

$$\text{f) } \sigma\left(\bigcap \left\{\hat{A} \mid \hat{A} \in \hat{\mathbf{A}}\right\}\right) \subseteq \bigcap \left\{\sigma(\hat{A}) \mid \hat{A} \in \hat{\mathbf{A}}\right\}.$$

If  $\hat{\mathbf{A}}$  is a collection of monads of subsets of  $\mathbb{R}$ , then

$$\sigma\left(\bigcap \left\{\hat{A} \mid \hat{A} \in \hat{\mathbf{A}}\right\}\right) = \bigcap \left\{\sigma(\hat{A}) \mid \hat{A} \in \hat{\mathbf{A}}\right\}.$$

Using **proposition 4.4**, **proposition 4.5**, and **proposition 4.6**, we could prove that the *monad* and *shadow operators* on subsets of  $\hat{\mathbb{R}}$  preserve the basic concepts of topology, and the concept of  $\sigma$ -algebra, which is fundamental in *Measure Theory*. This is clearly expressed in the next two propositions.

**Proposition 4.7 a)** If  $X \subseteq \mathbb{R}$ ,  $\mathbf{B}$  is a base for a topology for  $X$ , and  $\hat{\mathbf{B}} := \{m_{\approx}(A) \mid A \in \mathbf{B}\}$ , then

$\hat{\mathbf{B}}$  is a base for a topology for  $m_{\approx}(X)$ .

**b)** Let  $\mathbf{T}$  be a topology for  $\mathbb{R}$ .

If  $\hat{\mathbf{T}} := \{m_{\approx}(A) \mid A \in \mathbf{T}\}$ , then

$\hat{\mathbf{T}}$  is a topology for  $\hat{\mathbb{R}}$ .

**c)** If  $\mathbf{T}$  is a topology for  $\mathbb{R}$ ,  $\hat{\mathbf{T}} := \{m_{\approx}(A) \mid A \in \mathbf{T}\}$ , and  $X \subseteq \mathbb{R}$ , then

$$\text{int}_{\hat{\mathbf{T}}} m_{\approx}(X) = m_{\approx}(\text{int}_{\mathbf{T}} X),$$

$$\text{ext}_{\hat{\mathbf{T}}} m_{\approx}(X) = m_{\approx}(\text{ext}_{\mathbf{T}} X),$$

$$\text{bd}_{\hat{\mathbf{T}}} m_{\approx}(X) = m_{\approx}(\text{bd}_{\mathbf{T}} X),$$

$$\text{cl}_{\hat{\mathbf{T}}} m_{\approx}(X) = m_{\approx}(\text{cl}_{\mathbf{T}} X),$$

$$m_{\approx}(X) \text{ is open for } \hat{\mathbf{T}} \Leftrightarrow X \text{ is open for } \mathbf{T},$$

$$m_{\approx}(X) \text{ is closed for } \hat{\mathbf{T}} \Leftrightarrow X \text{ is closed for } \mathbf{T},$$

$$m_{\approx}(X) \text{ is compact for } \hat{\mathbf{T}} \Leftrightarrow X \text{ is compact for } \mathbf{T};$$

where  $\text{int}_{\hat{\mathbf{T}}}$ ,  $\text{int}_{\mathbf{T}}$ ,  $\text{ext}_{\hat{\mathbf{T}}}$ ,  $\text{ext}_{\mathbf{T}}$ ,  $\text{bd}_{\hat{\mathbf{T}}}$ ,  $\text{bd}_{\mathbf{T}}$ ,  $\text{cl}_{\hat{\mathbf{T}}}$ ,  $\text{cl}_{\mathbf{T}}$  are the *interior*, *exterior*, *boundary* and *closure* operators for the topologies  $\hat{\mathbf{T}}$  and  $\mathbf{T}$ , respectively.

**d)** If  $\mathbf{T}$  is a topology for  $\mathbb{R}$ ,  $\hat{\mathbf{T}} := \{m_{\approx}(A) \mid A \in \mathbf{T}\}$ ,  $X \subseteq \mathbb{R}$ , and  $Y \subseteq X$ , then

$$\hat{\mathbf{T}}_{m_{\approx}(X)} = \{m_{\approx}(A) \mid A \in \mathbf{T}_X\},$$

$$cl_{\widehat{\mathbf{T}}_{m_{\approx}(X)}} m_{\approx}(Y) = m_{\approx}(cl_{\mathbf{T}_X} Y);$$

where  $\widehat{\mathbf{T}}_{m_{\approx}(X)}$ ,  $\mathbf{T}_X$  are the relativizations of  $\widehat{\mathbf{T}}$ ,  $\mathbf{T}$  to  $m_{\approx}(X)$ ,  $X$ , respectively.

e) If  $\mathbf{T}$  is a topology for  $\mathbb{R}$ ,  $\widehat{\mathbf{T}} := \{m_{\approx}(A) | A \in \mathbf{T}\}$ , and  $X \subseteq \mathbb{R}$ , then

$$m_{\approx}(X) \text{ is connected for } \widehat{\mathbf{T}} \Leftrightarrow X \text{ is connected for } \mathbf{T}.$$

f) Let  $\mathbf{B}$  be a  $\sigma$ -algebra of subsets of  $\mathbb{R}$ .

If  $\widehat{\mathbf{B}} := \{m_{\approx}(A) | A \in \mathbf{B}\}$ , then

$$\widehat{\mathbf{B}} \text{ is a } \sigma\text{-algebra of subsets of } \widehat{\mathbb{R}}.$$

**Proposition 4.8** a) If  $\widehat{X} \subseteq \widehat{\mathbb{R}}$ ,  $\widehat{\mathbf{B}}$  is a base for a topology for  $\widehat{X}$  and a collection of monads of subsets of  $\mathbb{R}$ , and  $\mathbf{B} := \{\sigma(\widehat{A}) | \widehat{A} \in \widehat{\mathbf{B}}\}$ , then

$$\mathbf{B} \text{ is a base for a topology for } \sigma(\widehat{X}).$$

b) Let  $\widehat{\mathbf{T}}$  be a topology for  $\widehat{\mathbb{R}}$  and a collection of monads of subsets of  $\mathbb{R}$ .

If  $\mathbf{T} := \{\sigma(\widehat{A}) | \widehat{A} \in \widehat{\mathbf{T}}\}$ , then

$$\mathbf{T} \text{ is a topology for } \mathbb{R}.$$

c) If  $\widehat{\mathbf{T}}$  is a topology for  $\widehat{\mathbb{R}}$  and a collection of monads of subsets of  $\mathbb{R}$ ,  $\widehat{X}$  is the monad of a subset of  $\mathbb{R}$ , and  $\mathbf{T} := \{\sigma(\widehat{A}) | \widehat{A} \in \widehat{\mathbf{T}}\}$ , then

$$int_{\mathbf{T}} \sigma(\widehat{X}) = \sigma(int_{\widehat{\mathbf{T}}} \widehat{X}),$$

$$ext_{\mathbf{T}} \sigma(\widehat{X}) = \sigma(ext_{\widehat{\mathbf{T}}} \widehat{X}),$$

$$bd_{\mathbf{T}} \sigma(\widehat{X}) = \sigma(bd_{\widehat{\mathbf{T}}} \widehat{X}),$$

$$cl_{\mathbf{T}} \sigma(\widehat{X}) = \sigma(cl_{\widehat{\mathbf{T}}} \widehat{X}),$$

$$\sigma(\widehat{X}) \text{ is open for } \mathbf{T} \Leftrightarrow \widehat{X} \text{ is open for } \widehat{\mathbf{T}},$$

$$\sigma(\widehat{X}) \text{ is closed for } \mathbf{T} \Leftrightarrow \widehat{X} \text{ is closed for } \widehat{\mathbf{T}},$$

$$\sigma(\widehat{X}) \text{ is compact for } \mathbf{T} \Leftrightarrow \widehat{X} \text{ is compact for } \widehat{\mathbf{T}};$$

where  $int_{\widehat{\mathbf{T}}}$ ,  $int_{\mathbf{T}}$ ,  $ext_{\widehat{\mathbf{T}}}$ ,  $ext_{\mathbf{T}}$ ,  $bd_{\widehat{\mathbf{T}}}$ ,  $bd_{\mathbf{T}}$ ,  $cl_{\widehat{\mathbf{T}}}$ ,  $cl_{\mathbf{T}}$  are the *interior*, *exterior*, *boundary* and *closure* operators for the topologies  $\widehat{\mathbf{T}}$  and  $\mathbf{T}$ , respectively.

d) If  $\widehat{\mathbf{T}}$  is a topology for  $\widehat{\mathbb{R}}$  and a collection of monads of subsets of  $\mathbb{R}$ ,  $\widehat{Y} \subseteq \widehat{X} \subseteq \widehat{\mathbb{R}}$ ,

$\widehat{X}$  and  $\widehat{Y}$  are monads of subsets of  $\widehat{\mathbb{R}}$ , and  $\mathbf{T} := \left\{ \sigma(\widehat{A}) \mid \widehat{A} \in \widehat{\mathbf{T}} \right\}$ , then

$$\mathbf{T}_{\sigma(\widehat{X})} = \left\{ \sigma(\widehat{A}) \mid \widehat{A} \in \widehat{\mathbf{T}}_{\widehat{X}} \right\},$$

$$cl_{\mathbf{T}_{\sigma(\widehat{X})}} \sigma(\widehat{Y}) = \sigma(cl_{\widehat{\mathbf{T}}_{\widehat{X}}} \widehat{Y}) ;$$

where  $\widehat{\mathbf{T}}_{\widehat{X}}$ ,  $\mathbf{T}_{\sigma(\widehat{X})}$  are the relativizations of  $\widehat{\mathbf{T}}$ ,  $\mathbf{T}$  to  $\widehat{X}$ ,  $\sigma(\widehat{X})$ , respectively.

**e)** If  $\widehat{\mathbf{T}}$  is a topology for  $\widehat{\mathbb{R}}$  and a collection of monads of subsets of  $\mathbb{R}$ ,  $\widehat{X}$  is the monad of a subset of  $\mathbb{R}$ , and  $\mathbf{T} := \left\{ \sigma(\widehat{A}) \mid \widehat{A} \in \widehat{\mathbf{T}} \right\}$ , then

$$\sigma(\widehat{X}) \text{ is connected for } \mathbf{T} \Leftrightarrow \widehat{X} \text{ is connected for } \widehat{\mathbf{T}}.$$

**f)** Let  $\widehat{\mathbf{B}}$  be a  $\sigma$ -algebra of subsets of  $\widehat{\mathbb{R}}$  and a collection of monads of subsets of  $\mathbb{R}$ .

If  $\mathbf{B} := \left\{ \sigma(\widehat{A}) \mid \widehat{A} \in \widehat{\mathbf{B}} \right\}$ , then

$\mathbf{B}$  is a  $\sigma$ -algebra of subsets of  $\mathbb{R}$ .

## 5 The Derivative

Throughout this section, we shall not use any concept of *limit* in the generalized real continuum (i.e.  $\widehat{\mathbb{R}}$ ), working instead, in an actual manner, with the concepts of *indiscernibility*, *shadow*, *differential*, and *monad*. The concept of *limit* is only used in the *Cantor-Dedekind continuum* (i.e.  $\mathbb{R}$ ).

The first important step is the introduction of the concept of *indiscernible extension* of a function  $\phi : X \rightarrow Y$ , where  $X, Y \subseteq \mathbb{R}$ .

**Definition 5.1** Let  $X, Y \subseteq \mathbb{R}$ .

If  $\phi : X \rightarrow Y$  and  $f : m_{\approx}(X) \rightarrow m_{\approx}(Y)$  are functions, then  $f$  is said to be an *indiscernible extension* of  $\phi$  iff

$$(\forall x \in m_{\approx}(X))(f(\sigma x) = \phi(\sigma x) \wedge f(x) \approx \phi(\sigma x)).$$

Clearly:

**Proposition 5.2** Let  $X, Y \subseteq \mathbb{R}$ .

If  $\phi : X \rightarrow Y$ ,  $\psi : X \rightarrow Y$ ,  $f : m_{\approx}(X) \rightarrow m_{\approx}(Y)$  are functions, and  $f$  is an indiscernible extension of  $\phi$  and  $\psi$ , then

$$\phi = \psi.$$

Before introducing the concept of *interval* in  $\widehat{\mathbb{R}}$ , we must define the analogue on  $\widehat{\mathbb{R}}$  of the usual linear ordering  $\leq$  on  $\mathbb{R}$ .

**Definition 5.3** Let  $x, y \in \widehat{\mathbb{R}}$ .

We say that  $x$  is *less than or indiscernible from*  $y$ , and we denote it by  $x \lesssim y$ , iff  $\sigma x \leq \sigma y$  (where in  $\sigma x \leq \sigma y$  we consider the usual linear ordering  $\leq$  on  $\mathbb{R}$ ), and we say that  $x$  is *greater than or indiscernible from*  $y$ , and we denote it by  $x \gtrsim y$ , iff  $y \lesssim x$ .  $\widehat{\mathbb{R}}_0^+$  and  $\widehat{\mathbb{R}}_0^-$  denote the subsets of  $\widehat{\mathbb{R}}$  defined by

$$\widehat{\mathbb{R}}_0^+ := \{x \in \widehat{\mathbb{R}} \mid x \gtrsim 0\} = \widehat{\mathbb{R}}^+ \cup m_{\approx}(0),$$

$$\widehat{\mathbb{R}}_0^- := \{x \in \widehat{\mathbb{R}} \mid x \lesssim 0\} = \widehat{\mathbb{R}}^- \cup m_{\approx}(0).$$

Clearly:

**Proposition 5.4 a)** If  $x, y \in \widehat{\mathbb{R}}$ , then

$$x \lesssim y \Leftrightarrow x < y \vee x \approx y.$$

**b)** Let  $x, y, z \in \widehat{\mathbb{R}}$ . Then:

$$x \lesssim x,$$

$$x \lesssim y \wedge y \lesssim x \Rightarrow x \approx y,$$

$$x \lesssim y \wedge y \lesssim z \Rightarrow x \lesssim z,$$

$$x \lesssim y \vee y \lesssim x,$$

$$x \lesssim y \Rightarrow x + z \lesssim y + z,$$

$$x \lesssim y \wedge z \gtrsim 0 \Rightarrow xz \lesssim yz.$$

So if we adopt the version of the usual *antisymmetry* expressed by the second formula above, then we may consider  $\lesssim$  a linear ordering on  $\widehat{\mathbb{R}}$ .

c) If  $\hat{\varepsilon}$  and  $\hat{\delta}$  are infinitesimals, then

$$\hat{\varepsilon} \lesssim \hat{\delta} \wedge \hat{\delta} \lesssim \hat{\varepsilon}.$$

d)  $\widehat{\mathbb{R}}_0^+$  and  $\widehat{\mathbb{R}}_0^-$  are the sets of *nonnegative* and *nonpositive* generalized real numbers, i.e.

$$\widehat{\mathbb{R}}_0^+ = \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}^-,$$

and

$$\widehat{\mathbb{R}}_0^- = \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}^+.$$

Furthermore:

$$\widehat{\mathbb{R}}_0^+ \cap \widehat{\mathbb{R}}_0^- = m_{\approx}(0).$$

The next definition introduce concepts that are adaptations to  $\lesssim$  (and  $\gtrsim$ ), on  $\widehat{\mathbb{R}}$ , of the usual notions for  $\leq$  (and  $\geq$ ), on  $\mathbb{R}$ .

**Definition 5.5** Let  $\widehat{A} \subseteq \widehat{\mathbb{R}}$ , and  $L, l \in \widehat{\mathbb{R}}$ . Then:

$L$  is a  $\lesssim$ -upper bound of  $\widehat{A}$  iff

$$\left( \forall x \in \widehat{A} \right) x \lesssim L.$$

$l$  is a  $\lesssim$ -lower bound of  $\widehat{A}$  iff

$$\left( \forall x \in \widehat{A} \right) x \gtrsim l$$

$\widehat{A}$  is  $\lesssim$ -bounded above iff  $\widehat{A}$  has a  $\lesssim$ -upper bound, and  $\widehat{A}$  is  $\lesssim$ -bounded below iff  $\widehat{A}$  has a  $\lesssim$ -lower bound.

$\widehat{A}$  is  $\lesssim$ -bounded iff  $\widehat{A}$  is  $\lesssim$ -bounded above and  $\lesssim$ -bounded below.

$\widehat{A}$  is  $\lesssim$ -unbounded iff  $\widehat{A}$  is not  $\lesssim$ -bounded.

$L$  is a  $\lesssim$ -maximum of  $\widehat{A}$  iff  $L \in \widehat{A}$  and  $L$  is a  $\lesssim$ -upper bound of  $\widehat{A}$ .

$l$  is a  $\lesssim$ -minimum of  $\widehat{A}$  iff  $l \in \widehat{A}$  and  $l$  is a  $\lesssim$ -lower bound of  $\widehat{A}$ .

$L$  is a  $\lesssim$ -supremum of  $\widehat{A}$  iff  $L$  is a  $\lesssim$ -minimum of  $\lesssim\text{-Up}(\widehat{A})$ , where  $\lesssim\text{-Up}(\widehat{A})$  is the set of all  $\lesssim$ -upper bounds of  $\widehat{A}$ .

$l$  is a  $\lesssim$ -infimum of  $\widehat{A}$  iff  $l$  is a  $\lesssim$ -maximum of  $\lesssim\text{-Lo}(\widehat{A})$ , where  $\lesssim\text{-Lo}(\widehat{A})$  is the set of all  $\lesssim$ -lower bounds of  $\widehat{A}$ .

**Proposition 5.6** Let  $\hat{A} \subseteq \hat{\mathbb{R}}$ , and  $L, L', l, l' \in \hat{\mathbb{R}}$ .

a) If  $L \approx L'$  and  $l \approx l'$ , then

$L$  is a  $\lesssim$ -upper bound of  $\hat{A}$  iff  $L'$  is a  $\lesssim$ -upper bound of  $\hat{A}$ ,

and

$l$  is a  $\lesssim$ -lower bound of  $\hat{A}$  iff  $l'$  is a  $\lesssim$ -lower bound of  $\hat{A}$ .

b)  $\lesssim\text{-}Up(\hat{A})$  and  $\lesssim\text{-}Lo(\hat{A})$  are monads of subsets of  $\mathbb{R}$ .

c) If  $L$  is a  $\lesssim$ -maximum of  $\hat{A}$ , then

$$L' \text{ is a } \lesssim\text{-maximum of } \hat{A} \Rightarrow L' \approx L.$$

Similarly, if  $l$  is a  $\lesssim$ -minimum of  $\hat{A}$ , then

$$l' \text{ is a } \lesssim\text{-minimum of } \hat{A} \Rightarrow l' \approx l.$$

If  $L$  is a  $\lesssim$ -maximum of  $\hat{A}$ , and  $\hat{A}$  is the monad of a subset of  $\mathbb{R}$ , then

$$L' \approx L \Rightarrow L' \text{ is a } \lesssim\text{-maximum of } \hat{A}.$$

Similarly, if  $l$  is a  $\lesssim$ -minimum of  $\hat{A}$ , and  $\hat{A}$  is the monad of a subset of  $\mathbb{R}$ , then

$$l' \approx l \Rightarrow l' \text{ is a } \lesssim\text{-minimum of } \hat{A}.$$

d) If  $L$  is a  $\lesssim$ -supremum of  $\hat{A}$ , then

$$L' \text{ is a } \lesssim\text{-supremum of } \hat{A} \Leftrightarrow L' \approx L.$$

Similarly, if  $l$  is a  $\lesssim$ -infimum of  $\hat{A}$ , then

$$l' \text{ is a } \lesssim\text{-infimum of } \hat{A} \Leftrightarrow l' \approx l.$$

**Proof** a) is trivial, since  $\sigma L = \sigma L'$  and  $\sigma l = \sigma l'$ .

b) Using a), we have:

$$m_{\approx}(\lesssim\text{-}Up(\hat{A})) = \lesssim\text{-}Up(\hat{A}).$$

Then, using **proposition 4.4 c)**:

$$\lesssim\text{-}Up(\hat{A}) = m_{\approx}(\sigma(\lesssim\text{-}Up(\hat{A}))).$$



Similarly, for  $\lesssim\text{-}Lo(\hat{A})$ .

c) Let  $L$  be a  $\lesssim$ -maximum of  $\hat{A}$ .

If  $L'$  is a  $\lesssim$ -maximum of  $\hat{A}$ , then, since  $L, L' \in \hat{A}$ ,

$$L \lesssim L' \wedge L' \lesssim L.$$

So, by **proposition 5.4 b)**,

$$L \approx L'.$$

Let  $\hat{A}$  be the monad of a subset of  $\mathbb{R}$ .

If  $L' \approx L$ , then, by **a)**,

$$L' \text{ is a } \lesssim\text{-upper bound of } \hat{A}.$$

On the other hand, since  $L \in \hat{A}$ ,  $L' \approx L$ , and  $\hat{A}$  is the monad of a subset of  $\mathbb{R}$ , we have:

$$L' \in \hat{A}.$$

So

$$L' \text{ is a } \lesssim\text{-maximum of } \hat{A}.$$

Similarly, for the concept of  $\lesssim$ -minimum.

**d)** follows directly from **b)** and **c)**. ■

We have just seen that the concepts of  $\lesssim$ -upper bound and  $\lesssim$ -lower bound are invariant under indiscernibility, and so are the concepts of  $\lesssim$ -supremum and  $\lesssim$ -infimum.

**Corollary 5.7** Let  $\hat{A} \subseteq \hat{\mathbb{R}}$ , and  $L, l \in \hat{\mathbb{R}}$ .

**a)** If  $L$  is a  $\lesssim$ -supremum of  $\hat{A}$ , then  $\sigma L$  is also a  $\lesssim$ -supremum of  $\hat{A}$ , and each  $\lesssim$ -supremum of  $\hat{A}$  has  $\sigma L$  as its shadow.

When  $l$  is a  $\lesssim$ -infimum of  $\hat{A}$ ,  $\sigma l$  is also a  $\lesssim$ -infimum of  $\hat{A}$ , and each  $\lesssim$ -infimum of  $\hat{A}$  has  $\sigma l$  as its shadow.

**b)** If  $L$  is a  $\lesssim$ -maximum of  $\hat{A}$  and  $\sigma L \in \hat{A}$ , then  $\sigma L$  is a  $\lesssim$ -maximum of  $\hat{A}$ , and each  $\lesssim$ -maximum of  $\hat{A}$  has  $\sigma L$  as its shadow.

When  $l$  is a  $\lesssim$ -minimum of  $\hat{A}$  and  $\sigma l \in \hat{A}$ , then  $\sigma l$  is a  $\lesssim$ -minimum of  $\hat{A}$ , and each  $\lesssim$ -minimum of  $\hat{A}$  has  $\sigma l$  as its shadow.

**Proof** **a)** and **b)** follow immediately from **proposition 5.6 d)**, and **proposition 5.6 a)**, **c)**, respectively. ■

**Definition 5.8** Let  $\hat{A} \subseteq \hat{\mathbb{R}}$ , and  $L, l \in \hat{\mathbb{R}}$ .

If  $L$  is a  $\lesssim$ -supremum of  $\hat{A}$ , then  $\sigma L$  is called the *real supremum* of  $\hat{A}$ .

Similarly, if  $l$  is a  $\lesssim$ -infimum of  $\hat{A}$ , then  $\sigma l$  is called the *real infimum* of  $\hat{A}$ .

If  $L$  is a  $\lesssim$ -maximum of  $\hat{A}$  and  $\sigma L \in \hat{A}$ , then  $\sigma L$  is said to be the *real maximum* of  $\hat{A}$ .

In a similar manner, if  $l$  is a  $\lesssim$ -minimum of  $\hat{A}$  and  $\sigma l \in \hat{A}$ , then  $\sigma l$  is said to be the *real minimum* of  $\hat{A}$ .

We denote the *real supremum*, the *real infimum*, the *real maximum*, and the *real minimum* of  $\hat{A}$  by  $\sup_r \hat{A}$ ,  $\inf_r \hat{A}$ ,  $\max_r \hat{A}$ , and  $\min_r \hat{A}$ , respectively.

Before presenting a *Completeness Property* for  $\hat{\mathbb{R}}$ , we need the following lemma:

**Lemma 5.9** Let  $\hat{A} \subseteq \hat{\mathbb{R}}$ , and  $L, l \in \hat{\mathbb{R}}$ .

**a)**  $L$  is a  $\lesssim$ -upper bound of  $\hat{A}$  iff  $\sigma L$  is an upper bound of  $\sigma(\hat{A})$ .

$l$  is a  $\lesssim$ -lower bound of  $\hat{A}$  iff  $\sigma l$  is a lower bound of  $\sigma(\hat{A})$ .

**b)**  $\sigma(\lesssim\text{-Up}(\hat{A})) = \text{Up}(\sigma(\hat{A}))$ , and  $\sigma(\lesssim\text{-Lo}(\hat{A})) = \text{Lo}(\sigma(\hat{A}))$ ; where  $\text{Up}(\sigma(\hat{A}))$  and  $\text{Lo}(\sigma(\hat{A}))$  are the sets of all *upper bounds* and *lower bounds* of  $\sigma(\hat{A})$ , respectively, for the usual linear ordering  $\leq$  on  $\mathbb{R}$ .

**c)**  $L$  is a  $\lesssim$ -maximum of  $\hat{A} \Rightarrow \sigma L = \max \sigma(\hat{A})$ .

$l$  is a  $\lesssim$ -minimum of  $\hat{A} \Rightarrow \sigma l = \min \sigma(\hat{A})$ .

If  $\hat{A}$  is the monad of a subset of  $\mathbb{R}$ , then

$$\sigma L = \max \sigma(\hat{A}) \Rightarrow L \text{ is a } \lesssim\text{-maximum of } \hat{A}.$$

Similarly, if  $\hat{A}$  is the monad of a subset of  $\mathbb{R}$ , then

$$\sigma l = \min \sigma(\hat{A}) \Rightarrow l \text{ is a } \lesssim\text{-minimum of } \hat{A}.$$

**d)**  $L$  is a  $\lesssim$ -supremum of  $\hat{A} \Leftrightarrow \sigma L = \sup \sigma(\hat{A})$ .

$l$  is a  $\lesssim$ -infimum of  $\hat{A} \Leftrightarrow \sigma l = \inf \sigma(\hat{A})$ .

**Proof** **a)** Clearly:

$$\begin{aligned} L \text{ is a } \lesssim\text{-upper bound of } \hat{A} &\Leftrightarrow (\forall x \in \hat{A}) \sigma x \leq \sigma L \Leftrightarrow \\ &\Leftrightarrow (\forall \xi \in \sigma(\hat{A})) \xi \leq \sigma L \Leftrightarrow \sigma L \text{ is an upper bound of } \sigma(\hat{A}). \end{aligned}$$

We may use a similar proof for the notion of  $\lesssim$ -lower bound.

**b)** For each  $x \in \hat{\mathbb{R}}$ , we have, using **a)**, and **proposition 4.4 b), c)**:

$$x \in \lesssim -Up(\hat{A}) \Leftrightarrow \sigma x \in Up(\sigma(\hat{A})) \Leftrightarrow x \in m_{\approx}(Up(\sigma(\hat{A}))).$$

So

$$\lesssim -Up(\hat{A}) = m_{\approx}(Up(\sigma(\hat{A}))).$$

Then, using **proposition 4.4 b), c)**,

$$\sigma(\lesssim -Up(\hat{A})) = \sigma(m_{\approx}(Up(\sigma(\hat{A})))) = Up(\sigma(\hat{A})).$$

Similarly, for  $\lesssim -Lo(\hat{A})$ .

**c)** If  $L$  is  $\lesssim$ -maximum of  $\hat{A}$ , then

$$L \text{ is a } \lesssim\text{-upper bound of } \hat{A},$$

and so, by **a)**,

$$\sigma L \text{ is an upper bound of } \sigma(\hat{A}).$$

On the other hand, we have, since  $L \in \hat{A}$ :

$$\sigma L \in \sigma(\hat{A}).$$

So

$$\sigma L = \max \sigma(\hat{A}).$$

Let  $\hat{A}$  be the monad of a subset of  $\mathbb{R}$ .

If  $\sigma L = \max \sigma(\hat{A})$ , then  $\sigma L$  is an upper bound of  $\sigma(\hat{A})$ , and so, by **a)**,  $L$  is a  $\lesssim$ -upper bound of  $\hat{A}$ .

On the other hand, since  $L \approx \sigma L$  and  $\sigma L \in \sigma(\hat{A})$ ,

$$L \in m_{\approx}(\sigma(\hat{A})).$$

But  $m_{\approx}(\sigma(\hat{A})) = m_{\approx}(\hat{A})$  (by **proposition 4.4 c)**), and  $m_{\approx}(\hat{A}) = \hat{A}$  (by **proposition 4.5 b)**).

So

$$L \in \hat{A}.$$

We have just proven that

$$L \text{ is a } \lesssim\text{-maximum of } \hat{A}.$$

Similarly, for the notion of  $\lesssim$ -minimum.

**d)** Using **b)**, **c)**, and **proposition 5.6 b)**, we have:

$$\begin{aligned} L \text{ is a } \lesssim \text{-supremum of } \hat{A} &\Leftrightarrow L \text{ is a } \lesssim \text{-minimum of } \lesssim \text{-}Up(\hat{A}) \Leftrightarrow \\ &\Leftrightarrow \sigma L = \min \sigma(\lesssim \text{-}Up(\hat{A})) \Leftrightarrow \sigma L = \min Up(\sigma(\hat{A})) \Leftrightarrow \sigma L = \sup \sigma(\hat{A}). \end{aligned}$$

Similarly, for the notion of  $\lesssim$ -infimum. ■

**Theorem 5.10 (The Completeness Property of  $\widehat{\mathbb{R}}$ )**

Let  $\hat{A}$  be a nonempty subset of  $\widehat{\mathbb{R}}$ .

**a)** If  $\hat{A}$  is  $\lesssim$ -bounded above, then there exists  $\sup_r \hat{A}$ .

**b)** If  $\hat{A}$  is  $\lesssim$ -bounded below, then there exists  $\inf_r \hat{A}$ .

**Proof a)** If  $\hat{A}$  is  $\lesssim$ -bounded above, then

$$\lesssim \text{-}Up(\hat{A}) \neq \emptyset.$$

So

$$\sigma(\lesssim \text{-}Up(\hat{A})) \neq \emptyset.$$

Then, by **lemma 5.9 b)**,

$$Up(\sigma(\hat{A})) \neq \emptyset.$$

Since  $\sigma(\hat{A}) \neq \emptyset$  (because  $\hat{A} \neq \emptyset$ ), we infer, using the **Completeness Property** of  $\mathbb{R}$ , that there exists  $\sup \sigma(\hat{A})$ .

Denoting  $\sup \sigma(\hat{A})$  by  $L$ , we have, using **lemma 5.9 d)**, and the fact that  $L \in \mathbb{R}$ :

$$L = \sup \sigma(\hat{A}) \Leftrightarrow L \text{ is a } \lesssim \text{-supremum of } \hat{A} \Rightarrow L = \sup_r \hat{A}.$$

**b)** admits a similar proof. ■

**Definition 5.11** Let  $\alpha, \alpha_1, \beta, \beta_1 \in \mathbb{R}$ ; with  $\alpha \leq \beta$ .

The *closed*, *open*, and *half-open intervals* determined by the ordered pair  $(\alpha, \beta)$ , denoted by  $\widehat{[\alpha, \beta]}$ ,  $\widehat{] \alpha, \beta]}$ ,  $\widehat{[\alpha, \beta[}$  and  $\widehat{] \alpha, \beta[}$ , respectively, are defined by:

$$\begin{aligned} \widehat{[\alpha, \beta]} &:= \left\{ x \in \widehat{\mathbb{R}} \mid \alpha \lesssim x \lesssim \beta \right\}, \\ \widehat{] \alpha, \beta[} &:= \left\{ x \in \widehat{\mathbb{R}} \mid \alpha < x < \beta \right\}, \end{aligned}$$

$$\widehat{]\alpha, \beta]} := \left\{ x \in \widehat{\mathbb{R}} \mid \alpha < x \lesssim \beta \right\},$$

$$\widehat{[\alpha, \beta[} := \left\{ x \in \widehat{\mathbb{R}} \mid \alpha \lesssim x < \beta \right\}.$$

The intervals just introduced are  $\lesssim$ -bounded sets.

We use the symbols  $-\infty$  and  $+\infty$  to introduce the intervals that are  $\lesssim$ -unbounded sets:

$$\widehat{[\alpha_1, +\infty[} := \left\{ x \in \widehat{\mathbb{R}} \mid \alpha_1 \lesssim x \right\},$$

$$\widehat{]\alpha_1, +\infty[} := \left\{ x \in \widehat{\mathbb{R}} \mid \alpha_1 < x \right\},$$

$$\widehat{]-\infty, \beta_1]} := \left\{ x \in \widehat{\mathbb{R}} \mid x \lesssim \beta_1 \right\},$$

$$\widehat{]-\infty, \beta_1[} := \left\{ x \in \widehat{\mathbb{R}} \mid x < \beta_1 \right\},$$

$$\widehat{]-\infty, +\infty[} := \widehat{\mathbb{R}}.$$

The next proposition admits a quite straightforward proof (in particular, **e**) follows easily from **proposition 4.4 b), c), proposition 4.7 e), proposition 5.12 a)**, and the well-known fact that *the connected subsets of  $\mathbb{R}$ , for the usual topology, are the intervals*).

**Proposition 5.12 a)** The intervals in  $\widehat{\mathbb{R}}$  are the monads of the correspondent intervals in  $\mathbb{R}$ , and the intervals in  $\mathbb{R}$  are the shadows of the correspondent intervals in  $\widehat{\mathbb{R}}$ ; for example, if  $\alpha, \alpha_1, \beta \in \mathbb{R}$ , and  $\alpha \leq \beta$ , then

$$\widehat{[\alpha, \beta]} = m_{\approx}([\alpha, \beta]),$$

$$[\alpha, \beta] = \sigma\left(\widehat{[\alpha, \beta]}\right),$$

$$\widehat{[\alpha_1, +\infty[} = m_{\approx}([\alpha_1, +\infty[),$$

$$[\alpha_1, +\infty[ = \sigma\left(\widehat{[\alpha_1, +\infty[}\right).$$

**b)** Let  $\alpha, \beta \in \mathbb{R}$ , with  $\alpha \leq \beta$ . Then:

$$\widehat{[\alpha, \beta]} \neq \emptyset,$$

$$\widehat{]\alpha, \beta[} = \emptyset \Leftrightarrow \alpha = \beta,$$

$$\widehat{]\alpha, \beta]} = \emptyset \Leftrightarrow \alpha = \beta,$$

$$\widehat{[\alpha, \beta[} = \emptyset \Leftrightarrow \alpha = \beta.$$

c) Let  $\alpha, \alpha', \alpha_1, \alpha'_1, \beta, \beta', \beta_1, \beta'_1 \in \mathbb{R}$ , with  $\alpha \leq \beta$  and  $\alpha' \leq \beta'$ . Then:

$$\begin{aligned} \widehat{[\alpha, \beta]} &= \widehat{[\alpha', \beta']} \Rightarrow (\alpha, \beta) = (\alpha', \beta'), \\ \alpha < \beta \wedge \left( \widehat{] \alpha, \beta[} = \widehat{] \alpha', \beta'_1[} \vee \widehat{] \alpha, \beta[} = \widehat{] \alpha', \beta'_1[} \vee \widehat{] \alpha, \beta[} = \widehat{] \alpha', \beta'_1[} \right) &\Rightarrow (\alpha, \beta) = (\alpha', \beta'), \\ \alpha = \beta \wedge \left( \widehat{] \alpha, \beta[} = \widehat{] \alpha', \beta'_1[} \vee \widehat{] \alpha, \beta[} = \widehat{] \alpha', \beta'_1[} \vee \widehat{] \alpha, \beta[} = \widehat{] \alpha', \beta'_1[} \right) &\Rightarrow \alpha' = \beta', \\ \widehat{[\alpha_1, +\infty[} = \widehat{[\alpha'_1, +\infty[} \vee \widehat{[\alpha_1, +\infty[} = \widehat{[\alpha'_1, +\infty[} &\Rightarrow \alpha_1 = \alpha'_1, \\ \widehat{]-\infty, \beta_1]} = \widehat{]-\infty, \beta'_1]} \vee \widehat{]-\infty, \beta_1]} = \widehat{]-\infty, \beta'_1]} &\Rightarrow \beta_1 = \beta'_1. \end{aligned}$$

Intervals of different kind are never identical, unless they are both the empty set; for example (still with  $\alpha, \alpha_1, \beta \in \mathbb{R}$ , and  $\alpha \leq \beta$ ),

$$\begin{aligned} \widehat{[\alpha, \beta]} &\neq \widehat{[\alpha, \beta[}, \\ \widehat{] \alpha, \beta[} &\neq \widehat{[\alpha_1, +\infty[}, \\ \widehat{[\alpha_1, +\infty[} &\neq \widehat{]-\infty, \beta_1]}, \\ \widehat{[\alpha, \beta[} = \widehat{] \alpha, \beta[} &\Leftrightarrow \widehat{[\alpha, \beta[} = \widehat{] \alpha, \beta[} = \emptyset. \end{aligned}$$

d) If  $\widehat{I}$  is an interval in  $\widehat{\mathbb{R}}$ , then

$$\begin{aligned} (\exists^1 I) \left( I \text{ is an interval in } \mathbb{R} \wedge \widehat{I} = m_{\approx}(I) \right), \\ m_{\approx}(\widehat{I}) &= \widehat{I}, \\ \sigma(\widehat{I}) &\subseteq \widehat{I}. \end{aligned}$$

e) Let  $\mathbf{T}$  be the usual topology for  $\mathbb{R}$ , and let  $\widehat{\mathbf{T}} := \{m_{\approx}(A) | A \in \mathbf{T}\}$ .

If  $\widehat{X}$  is the monad of a subset of  $\mathbb{R}$ , then

$$\widehat{X} \text{ is connected for } \widehat{\mathbf{T}} \Leftrightarrow \widehat{X} \text{ is an interval in } \widehat{\mathbb{R}}.$$

Now we may introduce the concept of *length* of a  $\lesssim$ -bounded interval in  $\widehat{\mathbb{R}}$  (notice how **proposition 5.12 b), c)** is relevant to the next definition).

**Definition 5.13** Let  $\alpha, \beta \in \mathbb{R}$ , and  $\alpha \leq \beta$ . If  $\widehat{I}$  is one of the intervals  $\widehat{[\alpha, \beta]}$ ,  $\widehat{] \alpha, \beta[}$ ,  $\widehat{[\alpha, \beta]}$ ,  $\widehat{[\alpha, \beta[}$ , then the *length* of  $\widehat{I}$ , denoted by  $l(\widehat{I})$ , is defined by:

$$l(\widehat{I}) := \beta - \alpha.$$

Clearly:

**Proposition 5.14** If  $\alpha \in \mathbb{R}$ , then

$$l(\widehat{]\alpha, \alpha[}) = l(\widehat{]\alpha, \alpha]}) = l(\widehat{[\alpha, \alpha[}) = l(\widehat{[\alpha, \alpha]}) = 0,$$

but

$$\widehat{]\alpha, \alpha[} = \widehat{]\alpha, \alpha]} = \widehat{[\alpha, \alpha[} = \emptyset,$$

and

$$\widehat{[\alpha, \alpha]} = m_{\approx}(\alpha).$$

**Remark 5.15** The intervals in  $\widehat{\mathbb{R}}$  have no clear-cut (i.e. pointlike) extremities.

For example, if  $\alpha, \beta, \gamma \in \mathbb{R}$  and  $\alpha < \beta < \gamma$ , then  $\widehat{[\alpha, \beta]}$ ,  $\widehat{[\beta, \gamma]}$  have  $m_{\approx}(\alpha)$ ,  $m_{\approx}(\beta)$  and  $m_{\approx}(\beta)$ ,  $m_{\approx}(\gamma)$  as extremities, respectively, and

$$\widehat{[\alpha, \beta]} \cap \widehat{[\beta, \gamma]} = m_{\approx}(\beta).$$

The intervals in  $\widehat{\mathbb{R}}$  are particularly fit to devise a model for the *flux of Time*:

A *stretch of Time* is an interval  $\widehat{[\alpha, \beta]}$  ( $\alpha, \beta \in \mathbb{R}$ ;  $\alpha < \beta$ ) whose members will be called *instants*.

Each *now* is the intersection of two adjacent stretches of Time, such as

$$\widehat{[\alpha, \beta]} \cap \widehat{[\beta, \gamma]} \quad (\alpha, \beta, \gamma \in \mathbb{R}; \alpha < \beta < \gamma).$$

So each *now* is the monad of an instant, and consequently, a set of *indiscernible instants* with the power of the continuum and length 0, since, for each  $\beta \in \widehat{\mathbb{R}}$ ,

$$|m_{\approx}(\beta)| = 2^{\aleph_0} \wedge l(m_{\approx}(\beta)) = l(\widehat{[\beta, \beta]}) = 0.$$

Also, being the intersection of two adjacent intervals, each *now* has a dual *past-future* nature.

This conception of Time is reminiscent of the ideas of the *Stoic philosophers* (especially Chrysippos) [12].

We now present the concept of *differentiability*.

**Definition 5.16** Let  $I$  be an open interval in  $\mathbb{R}$ , let  $\xi_0 \in I$ , and let  $\phi : I \rightarrow \mathbb{R}$  be a function.

If  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  is an indiscernible extension of  $\phi$ , then  $f$  is said to be *differentiable* at  $\xi_0$  iff there exists a real number  $\alpha$  such that

$$(\forall x \in m_{\approx}(\xi_0)) f(x) = \phi(\xi_0) + \alpha dx,$$

with the proviso that  $\alpha := \lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ , when such limit exists in  $\mathbb{R}$  (considering the usual definition of *limit*).

If  $J$  is an open subinterval (in  $\mathbb{R}$ ) of  $I$ , then  $f$  is said to be *differentiable* on  $m_{\approx}(J)$  iff  $f$  is differentiable at each  $\xi_0 \in J$ .

**Proposition 5.17** Let  $I$  be an open interval in  $\mathbb{R}$ , let  $\xi_0 \in I$ , and let  $\phi : I \rightarrow \mathbb{R}$  be a function.

If  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  is an indiscernible extension of  $\phi$ ,  $\alpha$  and  $\beta$  are real numbers, and

$$(\forall x \in m_{\approx}(\xi_0)) (f(x) = \phi(\xi_0) + \alpha dx \wedge f(x) = \phi(\xi_0) + \beta dx),$$

then

$$\alpha = \beta.$$

**Proof** If we choose  $x \in m_{\approx}(\xi_0)$  such that  $dx$  is the eventually null sequence  $(1, 0, 0, 0, \dots)$ , then the conclusion follows at once from  $\alpha dx = \beta dx$ , since

$$\alpha dx = \beta dx \Leftrightarrow (\alpha, 0, 0, 0, \dots) = (\beta, 0, 0, 0, \dots). \blacksquare$$

**Definition 5.18** With the notation and the conditions of **definition 5.16**, if  $f$  is differentiable at  $\xi_0$ , then  $\alpha$  is called the *derivative* of  $f$  at  $x$ , for each  $x \in m_{\approx}(\xi_0)$ , and we denote it by  $f'(x)$ .

**Remark 5.19** Let  $I$  be an open interval in  $\mathbb{R}$ , let  $\xi_0 \in I$ , and let  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  be an indiscernible extension of  $\phi : I \rightarrow \mathbb{R}$ .

If  $f$  is differentiable at  $\xi_0$ , then  $f'(x)$  exists (in  $\mathbb{R}$ ), for each  $x \in m_{\approx}(\xi_0)$ , and  $f'(x) = f'(\xi_0)$ . But the differentiability of  $f$  at  $\xi_0$  does not entail the existence of  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ , although if this is the case, then  $f'(\xi_0)$  coincides with this limit, by the proviso of **definition 5.16**.



As an example, let us consider the functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and  $f : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$ , defined by  $\phi(\xi) := |\xi|$  and  $f(x) := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x \in m_{\approx}(0) \\ -x, & \text{if } x < 0 \end{cases}$ , where  $|\cdot|$  denote the usual absolute value in  $\mathbb{R}$ . Clearly,  $f$  is an indiscernible extension of  $\phi$ , differentiable at  $\xi_0 = 0$  with  $f'(\xi_0) = 0$ , but  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ .

**Proposition 5.20** Let  $I$  be an open interval in  $\mathbb{R}$ , let  $\xi_0 \in I$ , and let  $\phi : I \rightarrow \mathbb{R}$  be a function.

If  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  is an indiscernible extension of  $\phi$ , and  $f$  is differentiable at  $\xi_0$ , then

$$(\forall x \in m_{\approx}(\xi_0)) f(x) = f(\xi_0) + f'(\xi_0)dx.$$

(Notice that we could have written

$$(\forall x \in m_{\approx}(\xi_0)) f(x) = f(\sigma x) + f'(x)dx,$$

since  $\sigma x = \xi_0$  and  $f'(x) = f'(\xi_0)$ , for each  $x \in m_{\approx}(\xi_0)$ ).

**Proof** Just remember that  $f(\xi_0) = \phi(\xi_0)$ . ■

**Proposition 5.20** expresses, in analytic terms, the *geometric idea* associated with the concept of *differentiability*. This idea was clearly expressed by G. W. Leibniz and G. de L'Hôpital (via Johann Bernoulli), and it is closely related to the use of nilpotent infinitesimals, as the Dutch theologian and mathematician B. Nieuwentijt first realized (around 1695):

*If  $f$  is differentiable at  $\xi_0$ , then the graph of  $f$  coincides locally (i.e. for infinitesimal increments of the argument around  $\xi_0$ ) with its tangent at the point  $(\xi_0, f(\xi_0))$ .*

The next lemma is necessary to establish the basic algebraic rules of derivation.

**Lemma 5.21** Let  $I$  be an open interval in  $\mathbb{R}$ , and let  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$ ,  $g : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  be indiscernible extensions of  $\phi : I \rightarrow \mathbb{R}$ ,  $\psi : I \rightarrow \mathbb{R}$ , respectively.

a) For fixed  $\alpha, \beta \in \mathbb{R}$ , if  $\phi(\xi) := \alpha\xi + \beta$ , then we may define  $f$  by

$$f(x) := \alpha x + \beta.$$

b)  $f + g$ ,  $fg$  are indiscernible extensions of  $\phi + \psi$ ,  $\phi\psi$ , respectively.

c) If  $\psi(\xi) \neq 0$ , for each  $\xi \in I$ , then

$$\frac{f}{g} \text{ is an indiscernible extension of } \frac{\phi}{\psi}.$$

d) For fixed  $m \in \mathbb{N}$ :

$$f^m \text{ is an indiscernible extension of } \phi^m.$$

If  $\phi(\xi) > 0$ , for each  $\xi \in I$ , and  $m > 1$ , then

$$\sqrt[m]{f} \text{ is an indiscernible extension of } \sqrt[m]{\phi}.$$

e) Let  $J$  be an open interval in  $\mathbb{R}$  such that  $\phi(I) \subseteq J$ , and let  $h : m_{\approx}(J) \rightarrow \widehat{\mathbb{R}}$  be an indiscernible extension of  $\theta : J \rightarrow \mathbb{R}$ . Then:

$$h \circ f \text{ is an indiscernible extension of } \theta \circ \phi.$$

f) If  $f$  is injective and  $m_{\approx}(\phi(I)) \subseteq f(m_{\approx}(I))$ , then  $\phi$  is also injective and

$$f^{-1} \text{ is an indiscernible extension of } \phi^{-1}.$$

**Proof** Only the proof of e) and f) has some difficulty.

e) First, we shall prove that  $h \circ f$  makes sense.

Let  $z \in m_{\approx}(I)$ .

Then, since  $\sigma z \in I$  (by **proposition 4.4 b), c)**) and  $\phi(I) \subseteq J$ , we have:

$$f(z) \approx \phi(\sigma z) \in J.$$

So

$$f(z) \in m_{\approx}(J).$$

We have proven that

$$f(m_{\approx}(I)) \subseteq m_{\approx}(J).$$

Now let  $x \in m_{\approx}(I)$ .

Then

$$(h \circ f)(\sigma x) = h(f(\sigma x)) = h(\phi(\sigma x)) = h(\sigma \phi(\sigma x)) = \theta(\sigma \phi(\sigma x)) = \theta(\phi(\sigma x)) = (\theta \circ \phi)(\sigma x).$$

On the other hand, since  $\phi(\sigma x) = \sigma f(x)$ , we have:

$$(h \circ f)(x) = h(f(x)) \approx \theta(\phi(\sigma x)) = (\theta \circ \phi)(\sigma x).$$

We have proven that

$$h \circ f \text{ is an indiscernible extension of } \theta \circ \phi.$$

f) If  $f$  is injective, then so is  $\phi$ , since  $\phi(\xi) = f(\xi)$ , for each  $\xi \in I$ .

Let  $z \in m_{\approx}(I)$ .

Then, since  $\sigma f(z) \in f(m_{\approx}(I))$  (because  $\sigma z \in I$ , by **proposition 4.4 b), c)**,  $I \subseteq m_{\approx}(I)$ , and  $\sigma f(z) = \phi(\sigma z) = f(\sigma z)$ , we have:

$$\begin{aligned} f^{-1}(\sigma f(z)) &= f^{-1}(f(\sigma z)) = \sigma z = \phi^{-1}(\phi(\sigma z)) = \phi^{-1}(\sigma f(z)), \\ f^{-1}(f(z)) &= z \approx \sigma z = \phi^{-1}(\phi(\sigma z)) = \phi^{-1}(\sigma f(z)). \end{aligned}$$

Since  $f$  is an indiscernible extension of  $\phi$ , we have  $f(m_{\approx}(I)) \subseteq m_{\approx}(\phi(I))$ . So, from  $m_{\approx}(\phi(I)) \subseteq f(m_{\approx}(I))$ , we infer that

$$f(m_{\approx}(I)) = m_{\approx}(\phi(I)).$$

We have proven that

$$f^{-1} \text{ is an indiscernible extension of } \phi^{-1}. \blacksquare$$

Let us state the *basic algebraic properties of the derivative*:

**Proposition 5.22** Let  $I$  be an open interval in  $\mathbb{R}$ , let  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$ ,  $g : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  be indiscernible extensions of  $\phi : I \rightarrow \mathbb{R}$ ,  $\psi : I \rightarrow \mathbb{R}$ , respectively, and let  $\xi_0 \in I$ .

a) If  $\alpha$  and  $\beta$  are fixed real numbers, and  $f$  is defined by  $f(x) := \alpha x + \beta$ , then  $f$  is differentiable at  $\xi_0$ , and

$$(\forall x \in m_{\approx}(\xi_0)) f'(x) = \alpha.$$

b) Let  $f$  and  $g$  be differentiable at  $\xi_0$ . Then:

If at least one of the limits  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$  exists in  $\mathbb{R}$ , then  $f + g$  is differentiable at  $\xi_0$ , and for each  $x \in m_{\approx}(\xi_0)$ :

$$(f + g)'(x) = f'(x) + g'(x).$$

c) Let  $f$  and  $g$  be differentiable at  $\xi_0$ .

If  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  and  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$  exist in  $\mathbb{R}$ , then  $fg$  is differentiable at  $\xi_0$ , and we have, for each  $x \in m_{\approx}(\xi_0)$ :

$$(fg)'(x) = f'(x)g(\xi_0) + g'(x)f(\xi_0).$$

If  $\phi(\xi_0) \neq 0$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \psi(\xi)$  exist and  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ , then  $fg$  is differentiable at  $\xi_0$ , and we have, for each  $x \in m_{\approx}(\xi_0)$ :

$$(fg)'(x) = f'(x)g(\xi_0) + g'(x)f(\xi_0).$$

If  $\psi(\xi_0) \neq 0$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \phi(\xi)$  exist and  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  does not

exist in  $\mathbb{R}$ , then  $fg$  is differentiable at  $\xi_0$ , and we have, for each  $x \in m_{\approx}(\xi_0)$ :

$$(fg)'(x) = f'(x)g(\xi_0) + g'(x)f(\xi_0).$$

**d)** Let  $f$  and  $g$  be differentiable at  $\xi_0$ , and let  $\psi(\xi) \neq 0$ , for each  $\xi \in I$ .

If  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  and  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$  exist in  $\mathbb{R}$ , then  $\frac{f}{g}$  is differentiable at  $\xi_0$ , and we have, for each  $x \in m_{\approx}(\xi_0)$ :

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(\xi_0) - g'(x)f(\xi_0)}{g(\xi_0)^2}.$$

If  $\phi(\xi_0) \neq 0$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{1}{\psi(\xi)}$  exist and  $\lim_{\xi \rightarrow \xi_0} \frac{\frac{1}{\psi(\xi)} - \frac{1}{\psi(\xi_0)}}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ , then  $\frac{f}{g}$  is differentiable at  $\xi_0$ , and we have, for each  $x \in m_{\approx}(\xi_0)$ :

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(\xi_0) - g'(x)f(\xi_0)}{g(\xi_0)^2}.$$

If  $\lim_{\xi \rightarrow \xi_0} \frac{\frac{1}{\psi(\xi)} - \frac{1}{\psi(\xi_0)}}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \phi(\xi)$  exist and  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ , then  $\frac{f}{g}$  is differentiable at  $\xi_0$ , and for each  $x \in m_{\approx}(\xi_0)$ :

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(\xi_0) - g'(x)f(\xi_0)}{g(\xi_0)^2}.$$

**e)** Let  $m \in \mathbb{N}$ , and let  $f$  be differentiable at  $\xi_0$ .

If  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  exists in  $\mathbb{R}$ , then  $f^m$  is differentiable at  $\xi_0$ , and for each  $x \in m_{\approx}(\xi_0)$ :

$$(f^m)'(x) = mf(\xi_0)^{m-1}f'(x).$$

If  $\phi$  is continuous at  $\xi_0$  (considering the usual definition of *continuity* at a point),  $\phi(\xi_0) \neq 0$ , and  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ , then  $f^m$  is differentiable at  $\xi_0$ , and for each  $x \in m_{\approx}(\xi_0)$ :

$$(f^m)'(x) = mf(\xi_0)^{m-1}f'(x).$$

**f)** For fixed  $m \in \mathbb{N}$ , let  $f$  be differentiable at  $\xi_0$ , and let  $\phi(\xi) > 0$ , for each  $\xi \in I$ . Then  $\sqrt[m]{f}$  is differentiable at  $\xi_0$ , and for each  $x \in m_{\approx}(\xi_0)$ :

$$(\sqrt[m]{f})'(x) = \frac{f'(x)}{m \sqrt[m]{f(\xi_0)^{m-1}}}.$$

**Proof** This proposition is a straightforward consequence of **proposition 3.6** and **lemma 5.21**, except for the fact that we must be very careful with the proviso of **definition 5.16**. To illustrate the last point, we shall prove **c)**.

c) Let  $f$  and  $g$  be differentiable at  $\xi_0$ , and let  $x \in m_{\approx}(\xi_0)$ .

By **lemma 5.21 b)**,  $fg$  is an indiscernible extension of  $\phi\psi$ ; so we have, using **proposition 3.6 b)**:

$$(fg)(x) = (\phi\psi)(\xi_0) + d(fg)(x) = (\phi\psi)(\xi_0) + (f(\xi_0)g'(x) + g(\xi_0)f'(x))dx.$$

Before concluding that  $fg$  is differentiable at  $\xi_0$  and

$$(fg)'(x) = f(\xi_0)g'(x) + g(\xi_0)f'(x),$$

we must be very careful with the proviso of **definition 5.16**.

If  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$  exist in  $\mathbb{R}$ , then  $\lim_{\xi \rightarrow \xi_0} \frac{(\phi\psi)(\xi) - (\phi\psi)(\xi_0)}{\xi - \xi_0}$  also exists in  $\mathbb{R}$ , and equals  $f(\xi_0)g'(x) + g(\xi_0)f'(x)$ .

If  $\phi(\xi_0) \neq 0$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \psi(\xi)$  exist and  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ , then it is easy to prove that  $\lim_{\xi \rightarrow \xi_0} \frac{(\phi\psi)(\xi) - (\phi\psi)(\xi_0)}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ , and therefore the proviso is not violated.

When  $\psi(\xi_0) \neq 0$ ,  $\lim_{\xi \rightarrow \xi_0} \frac{\psi(\xi) - \psi(\xi_0)}{\xi - \xi_0}$ ,  $\lim_{\xi \rightarrow \xi_0} \phi(\xi)$  exist and  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  does not exist in  $\mathbb{R}$ , we may use the previous argument to obtain the same conclusion. ■

**Theorem 5.23 (Chain Rule)** Let  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$ ,  $g : m_{\approx}(J) \rightarrow \widehat{\mathbb{R}}$  be indiscernible extensions of  $\phi : I \rightarrow \mathbb{R}$ ,  $\psi : J \rightarrow \mathbb{R}$ , respectively, where  $I, J$  are open intervals in  $\mathbb{R}$  such that  $\phi(I) \subseteq J$ , and let  $\xi_0 \in I$ .

If  $f$  is differentiable at  $\xi_0$ ,  $g$  is differentiable at  $\eta_0 := f(\xi_0)$ , and both  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  and  $\lim_{\eta \rightarrow \eta_0} \frac{\psi(\eta) - \psi(\eta_0)}{\eta - \eta_0}$  exist in  $\mathbb{R}$ , then  $g \circ f$  is differentiable at  $\xi_0$ , and for each  $x \in m_{\approx}(\xi_0)$ :

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

**Proof** Let  $f$  be differentiable at  $\xi_0$ , and let  $g$  be differentiable at  $\eta_0 := f(\xi_0)$ .

By **lemma 5.21 e)**,  $g \circ f$  is an indiscernible extension of  $\psi \circ \phi$ ; so we have, for each  $x \in m_{\approx}(\xi_0)$ :

$$\begin{aligned} (g \circ f)(x) &= (\psi \circ \phi)(\xi_0) + d(g \circ f)(x) = \psi(\phi(\xi_0)) + d(g \circ f)(x) = \psi(f(\xi_0)) + d(g \circ f)(x) = \\ &= \psi(\eta_0) + d(g \circ f)(x) = g(\eta_0) + d(g \circ f)(x). \end{aligned}$$

On the other hand, since  $f$  is differentiable at  $\xi_0$ , and  $g$  is differentiable at  $\eta_0 = f(\xi_0)$ ,

$$(g \circ f)(x) = g(f(x)) = g(\eta_0 + f'(x)dx) = g(\eta_0) + g'(\eta_0)f'(x)dx.$$

By comparison with the previous result for  $(g \circ f)(x)$ , we infer that

$$d(g \circ f)(x) = g'(\eta_0)f'(x)dx.$$

Since  $f(x) \in m_{\approx}(\eta_0)$  (because  $f$  is differentiable at  $\xi_0$ ), and  $g$  is differentiable at  $\eta_0$ , we have:

$$d(g \circ f)(x) = g'(f(x))f'(x)dx..$$

And the proviso of **definition 5.16** is satisfied, since we obtain, as an immediate consequence of the usual **Chain Rule** in  $\mathbb{R}$  (and the differentiability of  $f, g$  at  $\xi_0, \eta_0$ , respectively) :

$$\begin{aligned} \lim_{\xi \rightarrow \xi_0} \frac{(\psi \circ \phi)(\xi) - (\psi \circ \phi)(\xi_0)}{\xi - \xi_0} &= \left( \lim_{\eta \rightarrow \eta_0} \frac{\psi(\eta) - \psi(\eta_0)}{\eta - \eta_0} \right) \left( \lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0} \right) = \\ &= g'(\eta_0)f'(\xi_0) = g'(f(x))f'(x). \end{aligned}$$

We have proven that  $g \circ f$  is differentiable at  $\xi_0$ , and for each  $x \in m_{\approx}(\xi_0)$ :

$$(g \circ f)'(x) = g'(f(x))f'(x). \blacksquare$$

**Theorem 5.24 (The Inverse Function Theorem)** Let  $I$  be an open interval in  $\mathbb{R}$ , let  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  be an injective indiscernible extension of a continuous function  $\phi : I \rightarrow \mathbb{R}$  (we consider the usual topology for  $\mathbb{R}$ , and its relativization to  $I$ ), and let  $\xi_0 \in I$ .

If  $m_{\approx}(\phi(I)) \subseteq f(m_{\approx}(I))$ ,  $f$  is differentiable at  $\xi_0$ ,  $f'(\xi_0) \neq 0$ , and  $\lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$  exists in  $\mathbb{R}$ , then  $f^{-1}$  (considered as a function with codomain  $\widehat{\mathbb{R}}$ ) is differentiable at  $\eta_0 := f(\xi_0)$ , and we have, for each  $y \in m_{\approx}(\eta_0)$ :

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

**Proof** Let  $J := \phi(I)$ ,  $\alpha := \lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ , and  $\eta_0 := f(\xi_0)$ .

Since  $\phi$  is continuous and injective (because  $f$  is an injective indiscernible extension of  $\phi$ ),  $\phi^{-1}$  is also continuous (considering the usual topology for  $\mathbb{R}$ , and its relativization to  $J$ ). So  $J = \phi(I) = (\phi^{-1})^{-1}(I)$  is an open interval in  $\mathbb{R}$ , and the same is valid for  $m_{\approx}(J)$  in  $\widehat{\mathbb{R}}$  (see **proposition 5.12 a**)).

As  $\alpha = f'(\xi_0) \neq 0$ , we know, by the usual **Inverse Function Theorem** in  $\mathbb{R}$ , that  $\beta := \lim_{\eta \rightarrow \eta_0} \frac{\phi^{-1}(\eta) - \phi^{-1}(\eta_0)}{\eta - \eta_0}$  exists in  $\mathbb{R}$ , and

$$\beta = \frac{1}{\alpha}.$$

Since  $m_{\approx}(J) = f(m_{\approx}(I))$  (see the proof of **lemma 5.21 f**)), we may consider the function  $g : m_{\approx}(J) \rightarrow \widehat{\mathbb{R}}$  defined by

$$g(y) := \begin{cases} f^{-1}(y), & \text{if } y \in m_{\approx}(J) \setminus m_{\approx}(\eta_0) \\ \phi^{-1}(\eta_0) + \beta dy, & \text{if } y \in m_{\approx}(\eta_0) \end{cases}.$$

Since, by **lemma 5.21 f)**,  $f^{-1}$  is an indiscernible extension of  $\phi^{-1}$ , to complete the prove we only need to show that  $g(y) = f^{-1}(y)$ , for each  $y \in m_{\approx}(\eta_0)$ .

If  $y \in m_{\approx}(\eta_0)$ , then  $g(y) \in m_{\approx}(\xi_0)$  (because  $\phi^{-1}(\eta_0) = f^{-1}(\eta_0) = \xi_0$ ), and since  $f$  is differentiable at  $\xi_0$ , we have:

$$f(g(y)) = f(\phi^{-1}(\eta_0) + \beta dy) = f(\xi_0 + \beta dy) = f(\xi_0) + \alpha \beta dy = \eta_0 + dy = y.$$

So

$$g(y) = f^{-1}(f(g(y))) = f^{-1}(y). \blacksquare$$

**Notation.** Let  $I$  be a nonempty open interval in  $\mathbb{R}$ , let  $\phi : I \rightarrow \mathbb{R}$  be a function, and let

$$A_{\phi} := \left\{ \xi_0 \in I \mid \lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0} \text{ exists in } \mathbb{R} \right\}.$$

The function  $\xi_0 \in A_{\phi} \mapsto \lim_{\xi \rightarrow \xi_0} \frac{\phi(\xi) - \phi(\xi_0)}{\xi - \xi_0}$ , from  $A_{\phi}$  to  $\mathbb{R}$ , will be denoted by  $\lambda_{\phi}$  (notice that we do not exclude, at least here, the case  $A_{\phi} = \emptyset$ ).

### Theorem 5.25 (The Mean Value Theorem)

Let  $I$  be a nonempty open interval in  $\mathbb{R}$ , let  $f : m_{\approx}(I) \rightarrow \widehat{\mathbb{R}}$  be an indiscernible extension of  $\phi : I \rightarrow \mathbb{R}$ , differentiable on  $m_{\approx}(I)$ , and let  $A_{\phi} = I$ .

If  $a, b \in m_{\approx}(I)$  and  $a < b$ , then there exists  $\gamma \in I$  such that  $a < \gamma < b$ , and

$$(1) \quad f(b) - f(a) = f'(\gamma)(b - a) + (f'(b) - f'(\gamma))db + (f'(\gamma) - f'(a))da.$$

In particular, if  $a, b \in I$ , then (1) assumes the familiar form:

$$f(b) - f(a) = f'(\gamma)(b - a).$$

The previous identities stay valid when we replace  $\gamma$  by any  $c \in m_{\approx}(\gamma)$ .

**Proof** Let  $a, b \in m_{\approx}(I)$ , and  $a < b$ .

Then

$$\sigma a < \sigma b.$$

So, by the usual **Mean Value Theorem**, there is  $\gamma \in I$  such that  $\sigma a < \gamma < \sigma b$ , and

$$\phi(\sigma b) - \phi(\sigma a) = \lambda_{\phi}(\gamma)(\sigma b - \sigma a).$$

Then, since  $f$  is an indiscernible extension of  $\phi$ , differentiable on  $m_{\approx}(I)$ , we have:

$$\begin{aligned} f(b) - f(a) &= \phi(\sigma b) + f'(b)db - \phi(\sigma a) - f'(a)da = f'(\gamma)(b - db - a + da) + \\ &+ f'(b)db - f'(a)da = f'(\gamma)(b - a) + (f'(b) - f'(\gamma))db + (f'(\gamma) - f'(a))da. \end{aligned}$$

Finally, by **definition 5.18**,  $f'(\gamma) = f'(c)$ , for each  $c \in m_{\approx}(\gamma)$ . ■

**Corollary 5.26** Let  $I$  be a nonempty open interval in  $\mathbb{R}$ , let  $f : m_{\approx}(I) \rightarrow \hat{\mathbb{R}}$  be an indiscernible extension of  $\phi : I \rightarrow \mathbb{R}$ , differentiable on  $m_{\approx}(I)$ , and let  $\Lambda_{\phi} = I$ .

- a) If  $f'(x) = 0$ , for each  $x \in m_{\approx}(I)$ , then  $f$  is a constant function.
- b) If  $f'(x) > 0$ , for each  $x \in m_{\approx}(I)$ , then  $f$  is a strictly increasing function.
- b) If  $f'(x) < 0$ , for each  $x \in m_{\approx}(I)$ , then  $f$  is a strictly decreasing function.

**Proof** a) Let  $a, b \in m_{\approx}(I)$ .

If  $a \approx b$ , then, since  $f$  is differentiable on  $m_{\approx}(I)$  with null derivative,

$$f(a) = \phi(\sigma a) + f'(a) da = \phi(\sigma a) = \phi(\sigma b) = \phi(\sigma b) + f'(b) db = f(b).$$

If  $a < b$  or  $b < a$ , then we obtain, as a direct consequence of identity (1) of **theorem 5.25**,

$$f(a) = f(b).$$

b) and c) admit trivial proofs, since if  $a, b \in m_{\approx}(I)$  and  $a < b$ , then we easily obtain, using identity (1) of **theorem 5.25**:

$$\sigma f(b) - \sigma f(a) = f'(\gamma)(\sigma b - \sigma a), \text{ for a certain } \gamma \text{ such that } a < \gamma < b. \blacksquare$$

We close this section with the introduction and elementary study of the concept of *natural indiscernible extension* of a function  $\phi : I \rightarrow \mathbb{R}$ , where  $I$  is a nonempty open interval in  $\mathbb{R}$ . *Natural indiscernible extensions* are the «natural» versions, in  $\hat{\mathbb{R}}$ , of the usual differentiable functions, in  $\mathbb{R}$ .

The starting point is the next proposition, which follows immediately from **definition 5.16** and **remark 5.19**.

**Proposition 5.27** Let  $I$  be a nonempty open interval in  $\mathbb{R}$ , and let  $\phi : I \rightarrow \mathbb{R}$  be a function such that  $\Lambda_{\phi} = I$ .

Then the function  $f : m_{\approx}(I) \rightarrow \hat{\mathbb{R}}$ , defined by  $f(x) := \phi(\sigma x) + \lambda_{\phi}(\sigma x)dx$ , is the unique indiscernible extension of  $\phi$  differentiable on  $m_{\approx}(I)$ .

**Definition 5.28** With the notation and the hypothesis of **proposition 5.27**, we call  $f : m_{\approx}(I) \rightarrow \hat{\mathbb{R}}$ , defined by  $f(x) := \phi(\sigma x) + \lambda_{\phi}(\sigma x)dx$ , the *natural indiscernible extension* of  $\phi$ , and we denote it by  $\hat{\phi}$ .



Natural indiscernible extensions preserve addition, scalar multiplication by a real number, multiplication, division, composition, and inversion, in a sense clearly expressed by **a)** to **e)**, and **g)**, in the next proposition.

**Proposition 5.29** Let  $I$  be a nonempty open interval in  $\mathbb{R}$ , and let  $\phi : I \rightarrow \mathbb{R}, \psi : I \rightarrow \mathbb{R}$  be functions such that  $\Lambda_\phi = \Lambda_\psi = I$ .

**a)**  $\Lambda_{\phi+\psi} = I$ , and  $\widehat{\phi + \psi} = \hat{\phi} + \hat{\psi}$ .

**b)** If  $\alpha \in \mathbb{R}$ , then  $\Lambda_{\alpha\phi} = I$ , and  $\widehat{\alpha\phi} = \alpha\hat{\phi}$ .

**c)**  $\Lambda_{\phi\psi} = I$ , and  $\widehat{\phi\psi} = \hat{\phi}\hat{\psi}$ .

**d)** If  $\psi(\xi) \neq 0$ , for each  $\xi \in I$ , then  $\Lambda_{\frac{\phi}{\psi}} = I$ , and

$$\widehat{\left(\frac{\phi}{\psi}\right)} = \frac{\hat{\phi}}{\hat{\psi}}.$$

**e)** If  $J$  is a nonempty open interval in  $\mathbb{R}$ ,  $\theta : J \rightarrow \mathbb{R}$  is a function such that  $\phi(I) \subseteq J$ , and  $\Lambda_\theta = J$ , then  $\Lambda_{\theta \circ \phi} = I$ , and

$$\widehat{\theta \circ \phi} = \hat{\theta} \circ \hat{\phi}.$$

**f)** If  $A$  is a nonempty subset of  $I$ , then

$$\hat{\phi}(m_\approx(A)) \subseteq m_\approx(\phi(A)).$$

If  $\lambda_\phi(\xi) \neq 0$ , for each  $\xi \in I$ , then

$$\hat{\phi}(m_\approx(A)) = m_\approx(\phi(A)).$$

If  $\alpha, \beta \in I, \alpha < \beta, \lambda_\phi(\xi) \neq 0$ , for each  $\xi \in ]\alpha, \beta[$ , and  $\lambda_\phi(\alpha) = \lambda_\phi(\beta) = 0$ , then

$$\hat{\phi}(\widehat{[\alpha, \beta]}) = m_\approx(\phi(]\alpha, \beta[)) \cup \{\phi(\alpha), \phi(\beta)\}.$$

**g)** If  $\phi$  is continuous, injective, and  $\lambda_\phi(\xi) \neq 0$ , for each  $\xi \in I$ , then  $\hat{\phi}$  is injective,  $\Lambda_{\phi^{-1}} = \phi(I)$ , and

$$\widehat{\phi^{-1}} = \hat{\phi}^{-1},$$

considering  $\phi^{-1}, \hat{\phi}^{-1}$  as functions with codomains  $\mathbb{R}, \hat{\mathbb{R}}$ , respectively.

**h)** If  $I = \mathbb{R}$  and  $\phi$  is an even function, then  $\hat{\phi}$  is also an even function, i.e.  $\hat{\phi}(-x) = \hat{\phi}(x)$ , for each  $x \in \hat{\mathbb{R}}$ .

Similarly, if  $I = \mathbb{R}$  and  $\phi$  is an odd function, then  $\hat{\phi}$  is also an odd function, i.e.  $\hat{\phi}(-x) = -\hat{\phi}(x)$ , for each  $x \in \hat{\mathbb{R}}$ .

i) If  $I = \mathbb{R}$ ,  $\lambda_0 \in \mathbb{R}^+$ , and  $\phi$  is a periodic function with period  $\lambda_0$ , then  $\hat{\phi}$  is also periodic with the same real period, i.e.

$$\lambda_0 = \min_r \left\{ l \in \widehat{\mathbb{R}}^+ \mid \left( \forall x \in \widehat{\mathbb{R}} \right) \hat{\phi}(x + l) = \hat{\phi}(x) \right\}.$$

**Proof a)** and **b)** admit trivial proofs, using the well-known identities (with different notation)  $\Lambda_{\phi+\psi} = \Lambda_{\alpha\phi} = I$ , and  $\lambda_{\phi+\psi} = \lambda_{\phi} + \lambda_{\psi}$ ,  $\lambda_{\alpha\phi} = \alpha\lambda_{\phi}$ .

**c)** Clearly,  $\Lambda_{\phi\psi} = I$ , and for each  $x \in m_{\approx}(I)$ , we have, using the well-known identity (with different notation)  $\lambda_{\phi\psi} = \lambda_{\phi}\psi + \lambda_{\psi}\phi$ :

$$\begin{aligned} \widehat{\phi\psi}(x) &= (\phi\psi)(\sigma x) + \lambda_{\phi\psi}(\sigma x)dx = \phi(\sigma x)\psi(\sigma x) + (\lambda_{\phi}(\sigma x)\psi(\sigma x) + \lambda_{\psi}(\sigma x)\phi(\sigma x))dx = \\ &= (\phi(\sigma x) + \lambda_{\phi}(\sigma x)dx)\psi(\sigma x) + \lambda_{\psi}(\sigma x)\phi(\sigma x)dx = \hat{\phi}(x)\psi(\sigma x) + (\psi(\sigma x) + \lambda_{\psi}(\sigma x)dx)\phi(\sigma x) - \\ &- \phi(\sigma x)\psi(\sigma x) = \hat{\phi}(x)\psi(\sigma x) + \hat{\psi}(x)\phi(\sigma x) - \phi(\sigma x)\psi(\sigma x) = \hat{\phi}(x)(\sigma\hat{\psi}(x)) + \hat{\psi}(x)(\sigma\hat{\phi}(x)) - \\ &- (\sigma\hat{\phi}(x))(\sigma\hat{\psi}(x)) = \hat{\phi}(x)\hat{\psi}(x) = (\hat{\phi}\hat{\psi})(x). \end{aligned}$$

**d)** Clearly  $\Lambda_{\frac{1}{\psi}} = I$ ; and for each  $x \in m_{\approx}(I)$ , we have, using the well-known identity (with different notation)  $\lambda_{\frac{1}{\psi}} = -\frac{\lambda_{\psi}}{\psi^2}$ :

$$\left( \widehat{\frac{1}{\psi}} \right)(x) = \frac{1}{\psi}(\sigma x) + \lambda_{\frac{1}{\psi}}(\sigma x)dx = \frac{1}{\psi(\sigma x)} - \frac{\lambda_{\psi}(\sigma x)}{\psi(\sigma x)^2}dx = \frac{\psi(\sigma x) - \lambda_{\psi}(\sigma x)dx}{\psi(\sigma x)^2}.$$

But (since the product of infinitesimals is always null)

$$\frac{\psi(\sigma x) - \lambda_{\psi}(\sigma x)dx}{\psi(\sigma x)^2}(\psi(\sigma x) + \lambda_{\psi}(\sigma x)dx) = \frac{\psi(\sigma x)^2 - \lambda_{\psi}(\sigma x)\psi(\sigma x)dx + \lambda_{\psi}(\sigma x)\psi(\sigma x)dx}{\psi(\sigma x)^2} = 1.$$

So

$$\frac{\psi(\sigma x) - \lambda_{\psi}(\sigma x)dx}{\psi(\sigma x)^2} = \frac{1}{\psi(\sigma x) + \lambda_{\psi}(\sigma x)dx} = \frac{1}{\hat{\psi}(x)} = \frac{1}{\hat{\psi}}(x).$$

We have proven that

$$\left( \widehat{\frac{1}{\psi}} \right) = \frac{1}{\hat{\psi}}.$$

Finally, using **c)**, we have:

$$\Lambda_{\frac{\phi}{\psi}} = \Lambda_{\phi\frac{1}{\psi}} = I,$$

and

$$\left( \widehat{\frac{\phi}{\psi}} \right) = \hat{\phi} \left( \widehat{\frac{1}{\psi}} \right) = \hat{\phi} \frac{1}{\hat{\psi}} = \frac{\hat{\phi}}{\hat{\psi}}.$$

**e)** Clearly,  $\Lambda_{\theta \circ \phi} = I$  ; and for each  $x \in m_{\approx}(I)$ , we have, using the well-known identity (with different notation)  $\lambda_{\theta \circ \phi} = (\lambda_{\theta} \circ \phi)\lambda_{\phi}$  :

$$\begin{aligned} \widehat{\theta \circ \phi}(x) &= (\theta \circ \phi)(\sigma x) + \lambda_{\theta \circ \phi}(\sigma x)dx = \theta(\phi(\sigma x)) + (\lambda_{\theta} \circ \phi)(\sigma x)\lambda_{\phi}(\sigma x)dx = \theta(\phi(\sigma x)) + \\ &+ \lambda_{\theta}(\phi(\sigma x))\lambda_{\phi}(\sigma x)dx = \theta(\phi(\sigma x)) + \lambda_{\theta}(\phi(\sigma x))d\hat{\phi}(x) = \theta(\sigma\hat{\phi}(x)) + \lambda_{\theta}(\sigma\hat{\phi}(x))d\hat{\phi}(x) = \\ &= \hat{\theta}(\hat{\phi}(x)) = (\hat{\theta} \circ \hat{\phi})(x). \end{aligned}$$

**f)** If  $x \in m_{\approx}(A)$ , then, since  $\sigma x \in A$  (by **proposition 4.4 b), c)**),

$$\hat{\phi}(x) = \phi(\sigma x) + \lambda_{\phi}(\sigma x)dx \in m_{\approx}(\phi(A)).$$

We have proven that

$$\hat{\phi}(m_{\approx}(A)) \subseteq m_{\approx}(\phi(A)).$$

Let  $\lambda_{\phi}(\xi) \neq 0$ , for each  $\xi \in I$  .

If  $\xi \in A$  and  $\hat{\varepsilon} \approx 0$ , then

$$\phi(\xi) + \hat{\varepsilon} = \phi(\xi) + \lambda_{\phi}(\xi)\frac{1}{\lambda_{\phi}(\xi)}\hat{\varepsilon} = \hat{\phi}\left(\xi + \frac{1}{\lambda_{\phi}(\xi)}\hat{\varepsilon}\right) \in \hat{\phi}(m_{\approx}(A)).$$

We have proven that

$$m_{\approx}(\phi(A)) \subseteq \hat{\phi}(m_{\approx}(A)).$$

Let  $\alpha, \beta \in I$ , with  $\alpha < \beta$ , let  $\lambda_{\phi}(\xi) \neq 0$ , for each  $\xi \in ]\alpha, \beta[$ , and let  $\lambda_{\phi}(\alpha) = \lambda_{\phi}(\beta) = 0$ .

We have:

$$\hat{\phi}(\widehat{[\alpha, \beta]}) = \hat{\phi}(\widehat{]\alpha, \beta[} \cup m_{\approx}(\alpha) \cup m_{\approx}(\beta)) = \hat{\phi}(\widehat{]\alpha, \beta[}) \cup \hat{\phi}(m_{\approx}(\alpha)) \cup \hat{\phi}(m_{\approx}(\beta)).$$

Since  $\lambda_{\phi}(\xi) \neq 0$ , for each  $\xi \in ]\alpha, \beta[$ , and  $\lambda_{\phi}(\alpha) = \lambda_{\phi}(\beta) = 0$ , we obtain (using the result we have just proven, and **definition 5.28**):

$$\hat{\phi}(\widehat{]\alpha, \beta[}) = \hat{\phi}(m_{\approx}(]\alpha, \beta[)) = m_{\approx}(\phi(]\alpha, \beta[)),$$

$$\hat{\phi}(m_{\approx}(\alpha)) = \{\phi(\alpha)\},$$

$$\hat{\phi}(m_{\approx}(\beta)) = \{\phi(\beta)\}.$$

So

$$\hat{\phi}(\widehat{[\alpha, \beta]}) = m_{\approx}(\phi(]\alpha, \beta[)) \cup \{\phi(\alpha), \phi(\beta)\}.$$

**g)** Clearly,  $\phi(I)$  is a nonempty open interval, and by the usual **Inverse Function Theorem**,  $\Lambda_{\phi^{-1}} = \phi(I)$  .

On the other hand, for each  $x_1, x_2 \in m_{\approx}(I)$ , we have (since  $\phi$  is injective and  $\lambda_{\phi}(\xi) \neq 0$ , for each  $\xi \in I$ ):

$$\begin{aligned} \hat{\phi}(x_1) = \hat{\phi}(x_2) &\Rightarrow \phi(\sigma x_1) + \lambda_{\phi}(\sigma x_1)dx_1 = \phi(\sigma x_2) + \lambda_{\phi}(\sigma x_2)dx_2 \Rightarrow \\ &\Rightarrow \begin{cases} \phi(\sigma x_1) = \phi(\sigma x_2) \\ \lambda_{\phi}(\sigma x_1)dx_1 = \lambda_{\phi}(\sigma x_2)dx_2 \end{cases} \Rightarrow \begin{cases} \sigma x_1 = \sigma x_2 \\ \lambda_{\phi}(\sigma x_1)(dx_1 - dx_2) = 0 \end{cases} \Rightarrow \\ &\Rightarrow \begin{cases} \sigma x_1 = \sigma x_2 \\ dx_1 = dx_2 \end{cases} \Rightarrow x_1 = x_2. \end{aligned}$$

So  $\hat{\phi}$  is also injective.

If  $x \in m_{\approx}(I)$ , then we have, using **e)** and denoting by  $\iota_I$  the inclusion function of  $I$  into  $\mathbb{R}$  :

$$\widehat{(\phi^{-1} \circ \hat{\phi})}(x) = \widehat{\phi^{-1} \circ \phi}(x) = \widehat{\iota_I}(x) = \iota_I(\sigma x) + \lambda_{\iota_I}(\sigma x)dx = \sigma x + dx = x.$$

If  $y \in m_{\approx}(\phi(I))$ , we have, using **e)** and denoting by  $\iota_{\phi(I)}$  the inclusion function of  $\phi(I)$  into  $\mathbb{R}$  :

$$(\hat{\phi} \circ \widehat{\phi^{-1}})(y) = \widehat{\phi \circ \phi^{-1}}(y) = \widehat{\iota_{\phi(I)}}(y) = \iota_{\phi(I)}(\sigma y) + \lambda_{\iota_{\phi(I)}}(\sigma y)dy = \sigma y + dy = y.$$

Finally, since the domains of  $\widehat{\phi^{-1}}$ ,  $\hat{\phi}^{-1}$  are  $m_{\approx}(\phi(I))$ ,  $\hat{\phi}(m_{\approx}(I))$ , and these sets are identical, by **f)**, we may consider proven that

$$\widehat{\phi^{-1}} = \hat{\phi}^{-1},$$

viewing  $\phi^{-1}$ ,  $\hat{\phi}^{-1}$  as functions with codomains  $\mathbb{R}$ ,  $\widehat{\mathbb{R}}$ , respectively.

**h)** admits a trivial proof, since

$$\phi \text{ is an even function} \Rightarrow \lambda_{\phi} \text{ is an odd function,}$$

$$\phi \text{ is an odd function} \Rightarrow \lambda_{\phi} \text{ is an even function,}$$

and  $d(-x) = -dx$ , for each  $x \in \widehat{\mathbb{R}}$ .

**i)** Let  $\lambda_0 \in \mathbb{R}^+$ , let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a periodic function with period  $\lambda_0$ , and let

$$\hat{L} := \left\{ l \in \widehat{\mathbb{R}}^+ \mid \left( \forall x \in \widehat{\mathbb{R}} \right) \hat{\phi}(x + l) = \hat{\phi}(x) \right\}.$$

For each  $x \in \widehat{\mathbb{R}}$ , we have, using the well-known fact that  $\lambda_{\phi}$  is also periodic with period  $\lambda_0$ :

$$\hat{\phi}(x + \lambda_0) = \phi(\sigma x + \lambda_0) + \lambda_{\phi}(\sigma x + \lambda_0)dx = \phi(\sigma x) + \lambda_{\phi}(\sigma x)dx = \hat{\phi}(x).$$

Then, since  $\mathbb{R}^+ \subseteq \widehat{\mathbb{R}}^+$ , we infer that

$$\lambda_0 \in \hat{L}.$$

On the other hand, if  $l \in \hat{L}$ , we have, for each  $\xi \in \mathbb{R}$  :

$$\phi(\xi + \sigma l) = \sigma \hat{\phi}(\xi + l) = \sigma \hat{\phi}(\xi) = \phi(\xi).$$

Then, since  $\sigma l \in \mathbb{R}^+$ ,

$$\lambda_0 \leq \sigma l.$$

So

$$\lambda_0 \lesssim l.$$

Since  $\lambda_0 \in \mathbb{R}$ ,  $\lambda_0 \in \hat{L}$ , and  $\lambda_0$  is an  $\lesssim$ -lower bound of  $\hat{L}$ , we conclude that

$$\lambda_0 = \min_r \hat{L}. \blacksquare$$

Frequently, physicists and engineers use identities like

$$(1 + dx)^\alpha = 1 + \alpha dx \quad (\text{for fixed } \alpha \in \mathbb{R}),$$

$$\sin(dx) = dx,$$

$$\cos(dx) = 1,$$

$$\exp(dx) = 1 + dx,$$

$$\log(1 + dx) = dx;$$

and they work with the functions involved in these identities as if they had the same basic properties as the usual ones. These procedures rely on powerful intuitions, but they are not rigorous and lead to contradictions in the framework of ordinary calculus. And yet they must be valid in a satisfactory calculus, based on an adequate (both for mathematics and the experimental sciences) generalization of the Cantor-Dedekind continuum. In the next example, we shall see how the *natural indiscernible extensions* give a positive answer to this aim, in the context of  $\hat{\mathbb{R}}$ .

**Example 5.30** Let  $I$  be a nonempty open interval in  $\mathbb{R}$ , and let  $\phi : I \rightarrow \mathbb{R}$  be a function such that  $A_\phi = I$ .

**1)** If  $\phi$  is a constant function, i.e.  $\phi(\xi) := \alpha$ , for each  $\xi \in I$ , where  $\alpha$  is a fixed real number, then, clearly, its natural indiscernible extension is also a constant function assuming the same value, i.e.  $\hat{\phi} : m_{\sim}(I) \rightarrow \hat{\mathbb{R}}$  is defined by

$$\hat{\phi}(x) := \alpha.$$

2) If  $\phi$  is the inclusion function of  $I$  into  $\mathbb{R}$ , i.e.  $\phi(\xi) := \xi$ , for each  $\xi \in I$ , then, since  $\lambda_\phi(\xi) = 1$  and  $\sigma x + dx = x$ , for each  $\xi \in \mathbb{R}$  and  $x \in m_\approx(I)$ , its natural indiscernible extension is the inclusion function of  $m_\approx(I)$  into  $\widehat{\mathbb{R}}$ , i.e.  $\hat{\phi} : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$  is defined by

$$\hat{\phi}(x) := x.$$

3) If  $\phi$  is a polynomial function, i.e.  $\phi(\xi) := \alpha_0 + \alpha_1 \xi + \dots + \alpha_m \xi^m$ , where  $\alpha_0, \alpha_1, \dots, \alpha_m$  are fixed real numbers, then, by the previous examples, **proposition 5.29 a), c)**, and mathematical induction, its natural indiscernible extension is also a polynomial function with the same coefficients, i.e.  $\hat{\phi} : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$  is defined by

$$\hat{\phi}(x) := \alpha_0 + \alpha_1 x + \dots + \alpha_m x^m.$$

4) If  $\phi$  is an algebraic function, i.e.  $\phi(\xi) := \frac{\psi(\xi)}{\theta(\xi)}$ , where  $\psi : I \rightarrow \mathbb{R}$ ,  $\theta : I \rightarrow \mathbb{R}$  are polynomial functions with real coefficients, and  $\theta(\xi) \neq 0$ , for each  $\xi \in I$ , then, by the last example and **proposition 5.29 d)**, its natural indiscernible extension is also an algebraic function, more precisely,  $\hat{\phi} : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$  is defined by

$$\hat{\phi}(x) := \frac{\hat{\psi}(x)}{\hat{\theta}(x)},$$

where  $\hat{\psi}$  and  $\hat{\theta}$  are the natural indiscernible extensions of  $\psi$  and  $\theta$ , respectively.

5) Let  $I := \mathbb{R}$ , and let  $\phi$  be the usual *exponential function*, denoted by  $\exp$ . Since  $\lambda_{\exp}(\xi) = \exp(\xi)$ , for each  $\xi \in \mathbb{R}$ , and  $m_\approx(\mathbb{R}) = \widehat{\mathbb{R}}$ , the natural indiscernible extension of  $\exp$  is the function  $\widehat{\exp} : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  defined by

$$\widehat{\exp}(x) := \exp(\sigma x) + \exp(\sigma x)dx.$$

$\widehat{\exp}$  has the same basic properties as  $\exp$ . For instance:

Using **proposition 5.29 f)**, we obtain:

$$\widehat{\exp}(\widehat{\mathbb{R}}) = \widehat{\exp}(m_\approx(\mathbb{R})) = m_\approx(\exp(\mathbb{R})) = m_\approx(\mathbb{R}^+) = \widehat{\mathbb{R}}^+.$$

If  $x \in \widehat{\mathbb{R}}$ , then

$$\widehat{\exp}'(x) = \exp(\sigma x) = \widehat{\exp}(\sigma x).$$

If  $x_1, x_2 \in \widehat{\mathbb{R}}$ , then (since  $dx_1 dx_2 = dx_2 dx_1 = 0$ )

$$\begin{aligned} \widehat{\exp}(x_1) \widehat{\exp}(x_2) &= (\exp(\sigma x_1) + \exp(\sigma x_1)dx_1)(\exp(\sigma x_2) + \exp(\sigma x_2)dx_2) = \\ &= \exp(\sigma(x_1 + x_2)) + \exp(\sigma(x_1 + x_2))d(x_1 + x_2) = \widehat{\exp}(x_1 + x_2). \end{aligned}$$

$\widehat{\exp}$  is a strictly increasing function, by **Corollary 5.26 b)**, since  $\widehat{\exp}'(x) = \exp(\sigma x) > 0$ , for each  $x \in \widehat{\mathbb{R}}$ .

And, of course,

$$\widehat{\exp}(0) = \exp(0) = 1.$$

$\widehat{\exp}$  is the adequate function for the afore mentioned considerations of physicists and engineers (as it is the case for the next examples of natural indiscernible extensions), since it has the basic properties of  $\exp$  and is defined not only for real numbers (where it assumes the same value as  $\exp$ ), but also for arguments involving infinitesimals. Moreover,  $\widehat{\exp}(x)$  is always indiscernible from  $\exp(\sigma x)$ .

Now we may infer, rigorously, that

$$\widehat{\exp}(dx) = \exp(\sigma(dx)) + \exp(\sigma(dx))dx = \exp(0) + \exp(0)dx = 1 + dx,$$

for each  $x \in \widehat{\mathbb{R}}$ .

**6)** Let  $I := \mathbb{R}$ , and let  $\phi$  be the usual *natural logarithm function*, which we denote by  $\log$ .

Since  $\lambda_{\log}(\xi) = \frac{1}{\xi}$ , for each  $\xi \in \mathbb{R}^+$ , and  $m_{\approx}(\mathbb{R}^+) = \widehat{\mathbb{R}}^+$ , the natural indiscernible extension of  $\log$  is the function  $\widehat{\log} : \widehat{\mathbb{R}}^+ \rightarrow \widehat{\mathbb{R}}$  defined by

$$\widehat{\log}(x) := \log(\sigma x) + \frac{1}{\sigma x}dx.$$

By **proposition 5.29 g)**, we have:

$$\widehat{\log} = \widehat{\exp}^{-1} = \widehat{\exp}^{-1}.$$

This result, in conjunction with the considerations of the previous example, suffices to assure that  $\widehat{\log}$  has the same basic properties as  $\log$ .

And since  $\widehat{\log} = \widehat{\exp}^{-1}$ , and  $\widehat{\exp}(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}^+$ , we have:

$$\widehat{\log}(\widehat{\mathbb{R}}^+) = \widehat{\mathbb{R}}.$$

Clearly,

$$\widehat{\log}'(x) = \frac{1}{\sigma x},$$

for each  $x \in \widehat{\mathbb{R}}^+$ .

Finally, we may infer, rigorously, that

$$\widehat{\log}(1+dx) = \log(\sigma(1+dx)) + \frac{1}{\sigma(1+dx)}dx = \log(1) + dx = dx,$$

for each  $x \in \widehat{\mathbb{R}}^+$ .

**7)** Let  $I := \mathbb{R}$ , and let  $\phi$  be the usual *sine function*, denoted by  $\sin$ .

Since  $\lambda_{\sin}(\xi) = \cos(\xi)$ , for each  $\xi \in \mathbb{R}$ , the natural indiscernible extension of  $\sin$  is the function  $\widehat{\sin} : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  defined by

$$\widehat{\sin}(x) := \sin(\sigma x) + \cos(\sigma x)dx.$$

Now let  $I := \mathbb{R}$ , and let  $\phi$  be the usual *cosine function*, denoted by  $\cos$ .

Since  $\lambda_{\cos}(\xi) = -\sin(\xi)$ , for each  $\xi \in \mathbb{R}$ , the natural indiscernible extension of  $\cos$  is the function  $\widehat{\cos} : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  defined by

$$\widehat{\cos}(x) := \cos(\sigma x) - \sin(\sigma x)dx.$$

$\widehat{\sin}$  and  $\widehat{\cos}$  have the same basic properties as  $\sin$  and  $\cos$ , respectively. For instance:

$\widehat{\sin}$  and  $\widehat{\cos}$  have real period  $2\pi$ , as it is clear from **proposition 5.29 i**).

Using the last result and **proposition 5.29 f**), we obtain:

$$\begin{aligned} \widehat{\sin}(\widehat{\mathbb{R}}) &= \widehat{\sin}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right) = \widehat{\sin}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) \cup \widehat{\sin}\left(\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right) = \\ &= m_{\approx}\left(\sin\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)\right) \cup \{-1, 1\} \cup m_{\approx}\left(\sin\left(\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right)\right) \cup \{-1, 1\} = ]-1, 1[ \cup \{-1, 1\}. \end{aligned}$$

Similarly,

$$\widehat{\cos}(\widehat{\mathbb{R}}) = ]-1, 1[ \cup \{-1, 1\}.$$

If  $x \in \widehat{\mathbb{R}}$ , then (since the square of an infinitesimal is always null)

$$\widehat{\sin}^2(x) = (\sin(\sigma x) + \cos(\sigma x)dx)^2 = \sin^2(\sigma x) + 2\sin(\sigma x)\cos(\sigma x)dx,$$

$$\widehat{\cos}^2(x) = (\cos(\sigma x) - \sin(\sigma x)dx)^2 = \cos^2(\sigma x) - 2\cos(\sigma x)\sin(\sigma x)dx.$$

So

$$\widehat{\sin}^2(x) + \widehat{\cos}^2(x) = \sin^2(\sigma x) + \cos^2(\sigma x) = 1.$$

If  $x_1, x_2 \in \widehat{\mathbb{R}}$ , then

$$\begin{aligned} \widehat{\sin}(x_1 \pm x_2) &= \sin(\sigma x_1 \pm \sigma x_2) + \cos(\sigma x_1 \pm \sigma x_2)d(x_1 \pm x_2) = \sin(\sigma x_1)\cos(\sigma x_2) \pm \\ &\quad \pm \sin(\sigma x_2)\cos(\sigma x_1) + (\cos(\sigma x_1)\cos(\sigma x_2) \mp \sin(\sigma x_1)\sin(\sigma x_2))d(x_1 \pm x_2). \end{aligned}$$

On the other hand (since the product of infinitesimals is always null),

$$\begin{aligned} \widehat{\sin}(x_1)\widehat{\cos}(x_2) &= (\sin(\sigma x_1) + \cos(\sigma x_1)dx_1)(\cos(\sigma x_2) - \sin(\sigma x_2)dx_2) = \\ &= \sin(\sigma x_1)\cos(\sigma x_2) + \cos(\sigma x_1)\cos(\sigma x_2)dx_1 - \sin(\sigma x_1)\sin(\sigma x_2)dx_2, \\ \widehat{\sin}(x_2)\widehat{\cos}(x_1) &= (\sin(\sigma x_2) + \cos(\sigma x_2)dx_2)(\cos(\sigma x_1) - \sin(\sigma x_1)dx_1) = \end{aligned}$$



$$= \sin(\sigma x_2) \cos(\sigma x_1) - \sin(\sigma x_1) \sin(\sigma x_2) dx_1 + \cos(\sigma x_1) \cos(\sigma x_2) dx_2.$$

So

$$\widehat{\sin}(x_1 \pm x_2) = \widehat{\sin}(x_1) \widehat{\cos}(x_2) \pm \widehat{\sin}(x_2) \widehat{\cos}(x_1).$$

In a similar manner, we could have proven that

$$\widehat{\cos}(x_1 \pm x_2) = \widehat{\cos}(x_1) \widehat{\cos}(x_2) \mp \widehat{\sin}(x_1) \widehat{\sin}(x_2).$$

And we clearly have, for each  $x \in \widehat{\mathbb{R}}$ :

$$\widehat{\sin}'(x) = \cos(\sigma x) = \widehat{\cos}(\sigma x),$$

$$\widehat{\cos}'(x) = -\sin(\sigma x) = -\widehat{\sin}(\sigma x).$$

Finally, we may infer, rigorously, that

$$\widehat{\sin}(dx) = \sin(\sigma(dx)) + \cos(\sigma(dx))dx = \sin(0) + \cos(0)dx = dx,$$

for each  $x \in \widehat{\mathbb{R}}$ .

Similarly,

$$\widehat{\cos}(dx) = \cos(\sigma(dx)) - \sin(\sigma(dx))dx = \cos(0) - \sin(0)dx = 1.$$

**8)** Let  $I := \mathbb{R}^+$ , let  $\alpha$  be a fixed real number, and let  $\phi$  be defined by  $\phi(\xi) := \xi^\alpha$ . Since  $\lambda_\phi(\xi) = \alpha \xi^{\alpha-1}$ , for each  $\xi \in \mathbb{R}^+$ , the natural indiscernible extension of  $\phi$  is the function  $\hat{\phi} : \widehat{\mathbb{R}}^+ \rightarrow \widehat{\mathbb{R}}$  defined by

$$\hat{\phi}(x) := (\sigma x)^\alpha + \alpha(\sigma x)^{\alpha-1} dx.$$

Clearly, for each  $x \in \widehat{\mathbb{R}}^+$ :

$$\hat{\phi}'(x) = \alpha(\sigma x)^{\alpha-1}.$$

If we denote  $\hat{\phi}(x)$  by  $x^\alpha$ , then

$$x^\alpha := (\sigma x)^\alpha + \alpha(\sigma x)^{\alpha-1} dx,$$

$$(x^\alpha)' = \alpha(\sigma x)^{\alpha-1},$$

for each  $x \in \widehat{\mathbb{R}}^+$ .

Trivially,  $\hat{\phi}(\widehat{\mathbb{R}}^+) = \{1\}$ , when  $\alpha = 0$ . If  $\alpha \neq 0$ , then we obtain, using **proposition 5.29 f)**:

$$\hat{\phi}(\widehat{\mathbb{R}}^+) = \hat{\phi}(m_\approx(\mathbb{R}^+)) = m_\approx(\phi(\mathbb{R}^+)) = m_\approx(\mathbb{R}^+) = \widehat{\mathbb{R}}^+.$$

As  $\phi(\xi) = \exp(\alpha \log(\xi))$ , for each  $\xi \in \mathbb{R}^+$ , we obtain, using the examples **1), 5), 6)**, and **proposition 5.29 c), e)**:

$$x^\alpha = \hat{\phi}(x) = \widehat{\exp}(\alpha \widehat{\log}(x)), \text{ for each } x \in \widehat{\mathbb{R}}^+.$$

Finally, we may infer, with complete rigour, that

$$(1 + dx)^\alpha = (\sigma(1 + dx))^\alpha + \alpha(\sigma(1 + dx))^{\alpha-1} dx = 1^\alpha + \alpha \cdot 1^{\alpha-1} dx = 1 + \alpha dx.$$

**9)** Let  $I := \mathbb{R}$ , let  $\alpha$  be a fixed positive real number, and let  $\phi$  be defined by  $\phi(\xi) := \alpha^\xi$ . Since  $\lambda_\phi(\xi) = \alpha^\xi \log(\alpha)$ , for each  $\xi \in \mathbb{R}$ , the natural indiscernible extension of  $\phi$  is the function  $\hat{\phi} : \hat{\mathbb{R}} \rightarrow \hat{\mathbb{R}}$  defined by

$$\hat{\phi}(x) := \alpha^{\sigma x} + \alpha^{\sigma x} \log(\alpha) dx.$$

Clearly, for each  $x \in \hat{\mathbb{R}}$ :

$$\hat{\phi}'(x) = \alpha^{\sigma x} \log(\alpha).$$

If we denote  $\hat{\phi}(x)$  by  $\alpha^x$ , then we have, for each  $x \in \hat{\mathbb{R}}$ :

$$\begin{aligned} \alpha^x &:= \alpha^{\sigma x} + \alpha^{\sigma x} \log(\alpha) dx, \\ (\alpha^x)' &= \alpha^{\sigma x} \log(\alpha). \end{aligned}$$

So, if  $e$  is *Euler's number*, then

$$e^x = e^{\sigma x} + e^{\sigma x} \log(e) dx = \exp(\sigma x) + \exp(\sigma x) dx = \widehat{\exp}(x),$$

for each  $x \in \hat{\mathbb{R}}$ .

Trivially,  $\hat{\phi}(\hat{\mathbb{R}}) = \{1\}$ , when  $\alpha = 1$ . If  $\alpha \neq 1$ , then we obtain, using **proposition 5.29 f)**:

$$\hat{\phi}(\hat{\mathbb{R}}) = \hat{\phi}(m_\approx(\mathbb{R})) = m_\approx(\phi(\mathbb{R})) = m_\approx(\mathbb{R}^+) = \hat{\mathbb{R}}^+.$$

The next definition introduces the concepts of *mth natural indiscernible extension* and *mth derivative function*, for  $m \in \mathbb{N}$ .

**Definition 5.31** Let  $I$  be a nonempty open interval in  $\mathbb{R}$ , and let  $\phi : I \rightarrow \mathbb{R}$  be a function such that  $\Lambda_\phi = I$ .

The functions  $\hat{\phi} : m_\approx(I) \rightarrow \hat{\mathbb{R}}$ ,  $\hat{\phi}' : m_\approx(I) \rightarrow \hat{\mathbb{R}}$  defined by

$$\begin{aligned} \hat{\phi}(x) &:= \phi(\sigma x) + \lambda_\phi(\sigma x) dx, \\ \hat{\phi}'(x) &:= \lambda_\phi(\sigma x), \end{aligned}$$

will be called the *first natural indiscernible extension* of  $\phi$ , and the *first derivative function* of  $\hat{\phi}$ , respectively. So the *first natural indiscernible extension* of  $\phi$  is, in fact, its *natural indiscernible extension*, and, most conveniently, the value of the *first*

derivative function of  $\hat{\phi}$  at  $\xi_0 \in I$  is its derivative at this point (see **definition 5.28** and **definition 5.18**, respectively).

If  $\Lambda_{\lambda_\phi} = I$ , then the functions  $\hat{\phi}^{[2]} : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$ ,  $\hat{\phi}'' : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$  defined by

$$\hat{\phi}^{[2]}(x) := \lambda_\phi(\sigma x) + \lambda_{\lambda_\phi}(\sigma x)dx,$$

$$\hat{\phi}''(x) := \lambda_{\lambda_\phi}(\sigma x),$$

will be called the *second natural indiscernible extension* of  $\phi$ , and the *second derivative function* of  $\hat{\phi}$ , respectively.

If  $\Lambda_{\lambda_{\lambda_\phi}} = I$ , then the functions  $\hat{\phi}^{[3]} : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$ ,  $\hat{\phi}''' : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$  defined by

$$\hat{\phi}^{[3]}(x) := \lambda_{\lambda_\phi}(\sigma x) + \lambda_{\lambda_{\lambda_\phi}}(\sigma x)dx,$$

$$\hat{\phi}'''(x) := \lambda_{\lambda_{\lambda_\phi}}(\sigma x),$$

will be called the *third natural indiscernible extension* of  $\phi$ , and the *third derivative function* of  $\hat{\phi}$ , respectively.

For the sake of uniformity, we also denote  $\hat{\phi}$ ,  $\hat{\phi}'$ ,  $\hat{\phi}''$ ,  $\hat{\phi}'''$  by  $\hat{\phi}^{[1]}$ ,  $\hat{\phi}^{(1)}$ ,  $\hat{\phi}^{(2)}$ ,  $\hat{\phi}^{(3)}$ , respectively.

We define in a similar manner the *fourth natural indiscernible extension* of  $\phi$  and the *fourth derivative function* of  $\hat{\phi}$ , denoted by  $\hat{\phi}^{[4]}$  and  $\hat{\phi}^{(4)}$ , respectively, . . . ; and if  $m \in \mathbb{N}$ , then we denote by  $\hat{\phi}^{[m]}$  and  $\hat{\phi}^{(m)}$  the *mth natural indiscernible extension* of  $\phi$  and the *mth derivative function* of  $\hat{\phi}$ , when such functions exist .

**Notation** Let  $m \in \mathbb{N}$ .

Under the conditions and with the notation of **definition 5.31**,  $\lambda_\phi^{(m)}$  will indicate that the symbol  $\lambda$  appears  $m$  times. For example:

$$\lambda_\phi^{(1)} := \lambda_\phi,$$

$$\lambda_\phi^{(2)} := \lambda_{\lambda_\phi},$$

$$\lambda_\phi^{(3)} := \lambda_{\lambda_{\lambda_\phi}}.$$

And if we define  $\lambda_\phi^{(0)} := \phi$ , then we have, for each  $x \in m_\approx(I)$ , and  $m \in \mathbb{N}$ :

$$\hat{\phi}^{[m]}(x) = \lambda_\phi^{(m-1)}(\sigma x) + \lambda_\phi^{(m)}(\sigma x)dx.$$

Since  $\lambda_\phi^{(0)} := \phi$ , it is «natural» to introduce the function  $\hat{\phi}^{(0)} : m_\approx(I) \rightarrow \widehat{\mathbb{R}}$ , defined by  $\hat{\phi}^{(0)}(x) := \phi(\sigma x) = \hat{\phi}(\sigma x)$ .

Clearly:

**Proposition 5.32** Let  $m \in \mathbb{N}$ . Then:

- a)  $\hat{\phi}^{[m]}$  is the (first) natural indiscernible extension of  $\lambda_{\phi}^{(m-1)}$ , i.e.  $\hat{\phi}^{[m]} = \widehat{\lambda_{\phi}^{(m-1)}}$ .
- b)  $\hat{\phi}^{(m)} = \sigma \circ \hat{\phi}^{[m+1]}$  (where  $\sigma : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  is the *shadow function*, i.e.  $\sigma(x) := \sigma x$ , for each  $x \in \widehat{\mathbb{R}}$ ).

**Remark 5.33** Let  $m \in \mathbb{N}$ .

If  $\hat{\phi}^{[m+1]}$  and  $\hat{\phi}^{(m+1)}$  exist, it is important to notice that  $\hat{\phi}^{(m+1)}$  is the derivative function of  $\hat{\phi}^{[m+1]}$ , and not the derivative function of  $\hat{\phi}^{(m)}$ . This is not surprising since  $\hat{\phi}^{[m+1]}$  is the (first) natural indiscernible extension of  $\lambda_{\phi}^{(m)}$ , and  $\lambda_{\phi}^{(m)}$  is, in fact, *the usual  $m$ th derivative function of  $\phi$* .

In blunt terms, the rule (valid for the *derivative at a point* or the *derivative function*) is

*The derivative is always associated with an indiscernible extension.*

Finally, it is important to realize that the range of  $\hat{\phi}^{(m)}$  is always a subset of  $\mathbb{R}$ , although its codomain is  $\widehat{\mathbb{R}}$ .

**Example 5.34 1)** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $\phi(\xi) := \xi^2$ . Then  $A_{\lambda_{\phi}^{(m)}} = \mathbb{R}$ , for each  $m \in \mathbb{N}_0$  (where  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ), and we have, for each  $\xi \in \mathbb{R}$ :

$$\begin{aligned}\lambda_{\phi}^{(0)}(\xi) &= \phi(\xi) = \xi^2, \\ \lambda_{\phi}^{(1)}(\xi) &= \lambda_{\phi}(\xi) = 2\xi, \\ \lambda_{\phi}^{(2)}(\xi) &= 2, \\ \lambda_{\phi}^{(m)}(\xi) &= 0, \text{ for } m \geq 3.\end{aligned}$$

Then, for each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ :

$$\hat{\phi}^{[m]}(x) = \lambda_{\phi}^{(m-1)}(\sigma x) + \lambda_{\phi}^{(m)}(\sigma x)dx = \begin{cases} (\sigma x)^2 + 2(\sigma x)dx = x^2, & \text{if } m = 1 \\ 2\sigma x + 2dx = 2x, & \text{if } m = 2 \\ 2, & \text{if } m = 3 \\ 0, & \text{if } m \geq 4 \end{cases},$$

as it should be, according to **example 5.30 3)**, and **proposition 5.32 a)**.

For each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ , we have:

$$\hat{\phi}^{(m)}(x) = \lambda_{\phi}^{(m)}(\sigma x) = \begin{cases} 2\sigma x, & \text{if } m = 1 \\ 2, & \text{if } m = 2 \\ 0, & \text{if } m \geq 3 \end{cases},$$

as it should be, according to the results we obtained for  $\hat{\phi}^{[m]}$ , and **proposition 5.32 b**).

We could have written the last identities more synthetically as

$$\begin{aligned}(x^2)' &= 2\sigma x, \\ (x^2)'' &= 2, \\ (x^2)^{(m)} &= 0, \text{ for } m \geq 3.\end{aligned}$$

**2)** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $\phi(\xi) := \exp(\xi)$ . Then  $A_{\lambda_\phi^{(m)}} = \mathbb{R}$ , for each  $m \in \mathbb{N}_0$ , and we have:

$$\lambda_\phi^{(m)}(\xi) = \exp(\xi), \text{ for each } \xi \in \mathbb{R}, \text{ and } m \in \mathbb{N}_0.$$

Then, for each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ :

$$\begin{aligned}\hat{\phi}^{[m]}(x) &= \lambda_\phi^{(m-1)}(\sigma x) + \lambda_\phi^{(m)}(\sigma x)dx = \exp(\sigma x) + \exp(\sigma x)dx = \widehat{\exp}(x), \\ \hat{\phi}^{(m)}(x) &= \lambda_\phi^{(m)}(\sigma x) = \exp(\sigma x) = \widehat{\exp}(\sigma x).\end{aligned}$$

More synthetically:

$$\begin{aligned}\widehat{\exp}^{[m]}(x) &= \widehat{\exp}(x), \\ \widehat{\exp}^{(m)}(x) &= \widehat{\exp}(\sigma x);\end{aligned}$$

for each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ .

**3)** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $\phi(\xi) := \sin(\xi)$ . Then  $A_{\lambda_\phi^{(m)}} = \mathbb{R}$ , for each  $m \in \mathbb{N}_0$ , and we have:

$$\lambda_\phi^{(m)}(\xi) = \begin{cases} (-1)^{\frac{m-1}{2}} \cos(\xi), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} \sin(\xi), & \text{if } m \text{ is even} \end{cases}.$$

Then, for each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ :

$$\begin{aligned}\hat{\phi}^{[m]}(x) &= \lambda_\phi^{(m-1)}(\sigma x) + \lambda_\phi^{(m)}(\sigma x)dx = \\ &= \begin{cases} (-1)^{\frac{m-1}{2}} \sin(\sigma x) + (-1)^{\frac{m-1}{2}} \cos(\sigma x) dx, & \text{if } m \text{ is odd} \\ (-1)^{\frac{m-2}{2}} \cos(\sigma x) + (-1)^{\frac{m}{2}} \sin(\sigma x) dx, & \text{if } m \text{ is even} \end{cases} = \\ &= \begin{cases} (-1)^{\frac{m-1}{2}} (\sin(\sigma x) + \cos(\sigma x)) dx, & \text{if } m \text{ is odd} \\ (-1)^{\frac{m-2}{2}} (\cos(\sigma x) - \sin(\sigma x)) dx, & \text{if } m \text{ is even} \end{cases} = \\ &= \begin{cases} (-1)^{\frac{m-1}{2}} \widehat{\sin}(x), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m-2}{2}} \widehat{\cos}(x), & \text{if } m \text{ is even} \end{cases}.\end{aligned}$$

For each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ , we have:

$$\begin{aligned}\widehat{\phi}^{(m)}(x) &= \lambda_{\phi}^{(m)}(\sigma x) = \\ &= \begin{cases} (-1)^{\frac{m-1}{2}} \cos(\sigma x), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} \sin(\sigma x), & \text{if } m \text{ is even} \end{cases} = \\ &= \begin{cases} (-1)^{\frac{m-1}{2}} \widehat{\cos}(\sigma x), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} \widehat{\sin}(\sigma x), & \text{if } m \text{ is even} \end{cases}.\end{aligned}$$

More synthetically, we have, for each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ :

$$\begin{aligned}\widehat{\sin}^{[m]}(x) &= \begin{cases} (-1)^{\frac{m-1}{2}} \widehat{\sin}(x), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m-2}{2}} \widehat{\cos}(x), & \text{if } m \text{ is even} \end{cases}, \\ \widehat{\sin}^{(m)}(x) &= \begin{cases} (-1)^{\frac{m-1}{2}} \widehat{\cos}(\sigma x), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} \widehat{\sin}(\sigma x), & \text{if } m \text{ is even} \end{cases}.\end{aligned}$$

For the *cosine function*, we have  $A_{\lambda_{\cos}^{(m)}} = \mathbb{R}$ , and  $\lambda_{\cos}^{(m)} = \lambda_{\lambda_{\sin}^{(m)}} = \lambda_{\sin}^{(m+1)}$ , for each  $m \in \mathbb{N}_0$ . Then, for each  $x \in \widehat{\mathbb{R}}$ , and  $m \in \mathbb{N}$ ,

$$\begin{aligned}\widehat{\cos}^{[m]}(x) &= \lambda_{\cos}^{(m-1)}(\sigma x) + \lambda_{\cos}^{(m)}(\sigma x)dx = \\ &= \lambda_{\sin}^{(m)}(\sigma x) + \lambda_{\sin}^{(m+1)}(\sigma x)dx = \widehat{\sin}^{(m)}(\sigma x) + \widehat{\sin}^{(m+1)}(\sigma x)dx = \\ &= \begin{cases} (-1)^{\frac{m-1}{2}} (\widehat{\cos}(\sigma x) - \widehat{\sin}(\sigma x)dx), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} (\widehat{\sin}(\sigma x) + \widehat{\cos}(\sigma x)dx), & \text{if } m \text{ is even} \end{cases} = \\ &= \begin{cases} (-1)^{\frac{m-1}{2}} \widehat{\cos}(x), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} \widehat{\sin}(x), & \text{if } m \text{ is even} \end{cases}; \\ \widehat{\cos}^{(m)}(x) &= \lambda_{\cos}^{(m)}(\sigma x) = \lambda_{\sin}^{(m+1)}(\sigma x) = \widehat{\sin}^{(m+1)}(\sigma x) = \begin{cases} (-1)^{\frac{m+1}{2}} \widehat{\sin}(\sigma x), & \text{if } m \text{ is odd} \\ (-1)^{\frac{m}{2}} \widehat{\cos}(\sigma x), & \text{if } m \text{ is even} \end{cases}.\end{aligned}$$

We close this section with **Taylor's Theorem**.

**Theorem 5.35 (Taylor's Theorem)** Let  $I$  be an open interval in  $\mathbb{R}$ , let  $\xi_0 \in I$  and  $m \in \mathbb{N}_0$ , and let  $\phi : I \rightarrow \mathbb{R}$  be a function such that  $A_{\lambda_{\phi}^{(k)}} = I$ , for each  $0 \leq k \leq m$ . Then for each  $x \in m_{\approx}(I \setminus \{\xi_0\})$  there exists a real number  $\theta \in ]0, 1[$  such that:

$$\widehat{\phi}(x) \approx \sum_{k=0}^m \frac{\widehat{\phi}^{(k)}(\xi_0)}{k!} (x - \xi_0)^k + \frac{(x - \xi_0)^{m+1}}{(m+1)!} \widehat{\phi}^{(m+1)}(\xi_0 + \theta(x - \xi_0)).$$

**Proof** By the usual **Taylor's Theorem** with the Lagrange form of the remainder, for each  $x \in m_{\approx}(I \setminus \{\xi_0\})$  there exists a real number  $\theta \in ]0, 1[$  such that we have:

$$\begin{aligned}\hat{\phi}(x) &\approx \phi(\sigma x) = \sum_{k=0}^m \frac{\lambda_{\phi}^{(k)}(\xi_0)}{k!} (\sigma x - \xi_0)^k + \frac{(\sigma x - \xi_0)^{m+1}}{(m+1)!} \lambda_{\phi}^{(m+1)}(\xi_0 + \theta(\sigma x - \xi_0)) \approx \\ &\approx \sum_{k=0}^m \frac{\hat{\phi}^{(k)}(\xi_0)}{k!} (x - \xi_0)^k + \frac{(x - \xi_0)^{m+1}}{(m+1)!} \hat{\phi}^{(m+1)}(\xi_0 + \theta(x - \xi_0)). \blacksquare\end{aligned}$$

## 6 The Differential Treatment of Singularities (two examples)

For each  $\xi_0 \in \mathbb{R}$ ,  $m_{\approx}(\xi_0)$  has three remarkable features:

(i) It has the same cardinality as  $\widehat{\mathbb{R}}$ , since (see **proposition 4.1** and its proof)

$$|m_{\approx}(\xi_0)| = |\widehat{\mathbb{R}}| = 2^{\aleph_0}.$$

(ii) It is a closed interval in  $\widehat{\mathbb{R}}$  with length 0, since (see **proposition 5.14**)

$$m_{\approx}(\xi_0) = [\widehat{\xi_0}, \widehat{\xi_0}],$$

$$l([\widehat{\xi_0}, \widehat{\xi_0}]) = 0.$$

In this sense,  $m_{\approx}(\xi_0)$  may be viewed as a *tiny* subset of  $\widehat{\mathbb{R}}$ .

(iii) It has a geometric structure, since (see **proposition 4.2 b**))

$m_{\approx}(\xi_0)$  is an infinite-dimensional real affine space.

We may use (ii) to obtain immediately:

(ii') If  $\xi_0 \in \mathbb{R}$ , then

$$m_{\approx}(\xi_0) \cap \mathbb{R} = \{\xi_0\}.$$

(i) and (iii) express properties of  $m_{\approx}(\xi_0)$  that are shared with the *entire* generalized real continuum (the fact that  $\widehat{\mathbb{R}}$  is an infinite-dimensional real affine space may be easily derived from **proposition 2.3 a**) and (iii)). Nevertheless  $m_{\approx}(\xi_0)$  is a *tiny* subset of  $\widehat{\mathbb{R}}$ , by (ii). This *global-local* nature of  $m_{\approx}(\xi_0)$  is the source of its usefulness for the *differential calculus*. In the next two examples, we apply this dual nature to the differential treatment of a singularity, using (ii') and (iii).

**Example 6.1 1)** Consider, in  $\mathbb{R}$ , the differential equation:

$$(2) \quad \xi'(\tau) = \begin{cases} -1, & \text{if } \tau < 0 \\ 1, & \text{if } \tau = 0 \\ 1, & \text{if } \tau > 0 \end{cases} .$$

Equation (2) has no solution on any open interval  $I$  in  $\mathbb{R}$  such that  $0 \in I$ , since if such a solution  $\xi : I \rightarrow \mathbb{R}$  existed, then  $\xi'$  would not satisfy the *intermediate value property* on  $I$  [see Fig. 1], violating **Darboux's Theorem**.

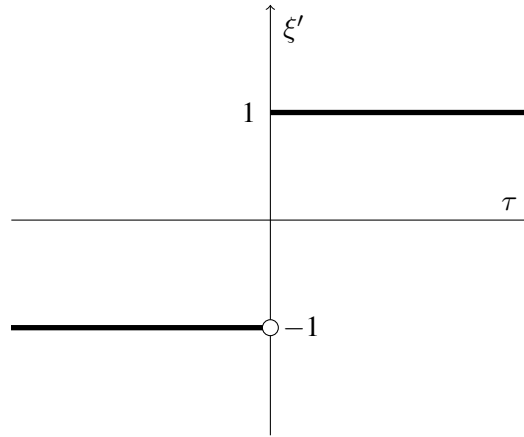


Fig. 1:  $\xi'$  would not satisfy the *intermediate value property* on  $I$ , for any open interval  $I$  in  $\mathbb{R}$  such that  $0 \in I$ .

Now consider the corresponding differential equation in  $\widehat{\mathbb{R}}$ :

$$(3) \quad x'(t) = \begin{cases} -1, & \text{if } t < 0 \\ 1, & \text{if } t \in m_{\approx}(0) \\ 1, & \text{if } t > 0 \end{cases} .$$

Equation (3) has an infinity of solutions on  $\widehat{\mathbb{R}}$ ; for instance, one solution is [see Fig. 2]

$$x(t) := \begin{cases} -t, & \text{if } t < 0 \\ dt, & \text{if } t \in m_{\approx}(0) \\ t, & \text{if } t > 0 \end{cases} = \begin{cases} -t, & \text{if } t < 0 \\ t, & \text{if } t \in m_{\approx}(0) \\ t, & \text{if } t > 0 \end{cases} .$$



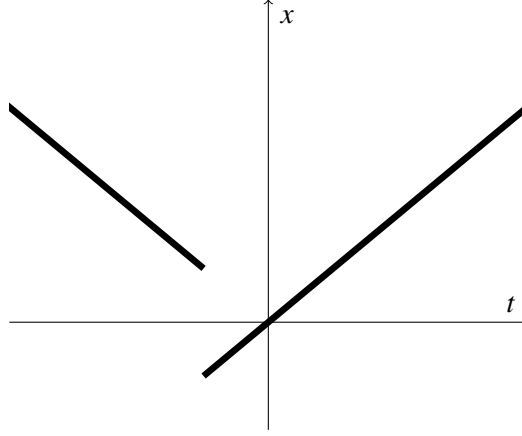


Fig. 2: A solution  $x : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  of the differential equation (3)

Notice that  $x : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  is an indiscernible extension of  $\xi : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $\xi(\tau) := |\tau|$ .

2) Consider, in  $\mathbb{R}$ , the differential equation:

$$(4) \quad \xi'(\tau) = \delta_0(\{\tau\}) = \begin{cases} 1, & \text{if } \tau = 0 \\ 0, & \text{if } \tau \neq 0 \end{cases}.$$

By **Darboux's Theorem**, equation (4) has no solution on any open interval  $I$  in  $\mathbb{R}$  such that  $0 \in I$  [see Fig. 3].

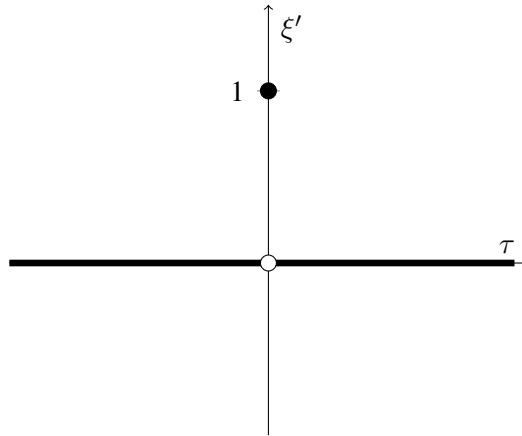


Fig. 3:  $\xi'$  would not satisfy the *intermediate value property* on  $I$ , for any open interval  $I$  in  $\mathbb{R}$  such that  $0 \in I$ .

Now consider the corresponding differential equation in  $\widehat{\mathbb{R}}$  :

$$(5) \quad x'(t) = \begin{cases} 1, & \text{if } t \in m_{\approx}(0) \\ 0, & \text{if } t \notin m_{\approx}(0) \end{cases} .$$

Equation (5) has an infinity of solutions on  $\widehat{\mathbb{R}}$ ; for instance, one solution is [see Fig. 4]

$$x(t) := \begin{cases} 1 + dt, & \text{if } t \in m_{\approx}(0) \\ 1, & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{cases} .$$

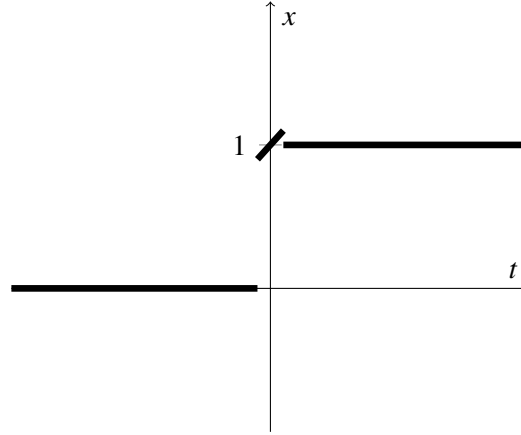


Fig. 4: A solution  $x : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  of the differential equation  $x'(t) = \begin{cases} 1, & \text{if } t \in m_{\approx}(0) \\ 0, & \text{if } t \notin m_{\approx}(0) \end{cases} .$

Notice that  $x : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$  is an indiscernible extension of the well-known *Heaviside*

*function*  $H : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $H(\tau) := \begin{cases} 1, & \text{if } \tau \geq 0 \\ 0, & \text{if } \tau < 0 \end{cases} .$

## 7 Conclusion

The purpose of this work was not to provide a tool to use the concept of *actual infinitesimal* as an alternative to the  $\varepsilon$ - $\delta$  definition of *limit*. In fact, we use the concept of *infinitesimal* (and the concepts of *shadow*, *monad*, *indiscernibility*) in the *mode of actuality* (in loose terms, the *mode of*  $\widehat{\mathbb{R}}$ , without a definition of *limit*), and the usual definition of *limit* in the *mode of potentiality* (in loose terms, the *mode of*  $\mathbb{R}$ ). It is our strong conviction that the *modes of actuality* and *potentiality* are both necessary (occasionally together, as in the definition of *differentiability*) to a Calculus suitable

not only for mathematicians, but also for experimental scientists. We must keep in mind that physicists and engineers need the concept of *limit*, and accept the usual  $\varepsilon$ - $\delta$  definition (though they use it as little as possible, as most mathematicians), but they also want to use the heuristic and computational power of *actual infinitesimal methods*.

Five other features of this work are worth mentioning:

- c<sub>1</sub>)** *The use of explicit actual infinitesimals.*
- c<sub>2</sub>)** *The local coincidence of the graph of a function  $f$ , differentiable at  $\xi_0 \in \mathbb{R}$ , with its tangent at  $(\xi_0, f(\xi_0))$ .*
- c<sub>3</sub>)** *The global-local nature of monads of points.*
- c<sub>4</sub>)** *The set-theoretic and topological properties of monads of subsets of  $\widehat{\mathbb{R}}$ .*
- c<sub>5</sub>)** *The sets we use are those of **ZFC** (Zermelo-Fraenkel Set Theory with the Axiom of Choice), without any distinction between internal and external sets.*

**c<sub>1</sub>)** is a positive answer to the uneasiness caused by the nonexplicit character of nonnull infinitesimals in *Non-standard Analysis* (see, for example, Alain Connes' criticism in [3], §2, p. 211).

We believe that a generalization of **c<sub>2</sub>)** is instrumental in *differential geometry*, especially for the definition of the *tangent space* to a manifold at a certain point.

**c<sub>3</sub>)** was already used in the differential treatment of some *singularities*, but we are convinced of its usefulness in the treatment of many others, in the area of *differential equations*. Moreover, the fact that  $m_{\approx}(0)$  contains the *real Hilbert space*  $l^2$  is very interesting since this space is isomorphic and isometric to any *separable real Hilbert space*.

As to **c<sub>4</sub>)**, the *set-theoretic* and *topological* properties of monads of subsets of  $\widehat{\mathbb{R}}$  seem to reveal a pattern extensible to other areas of mathematics.

**c<sub>5</sub>)** is a positive answer to one major difficulty encountered by non-standard analysts (especially those who work within the framework of *Internal Set Theory*): *external sets*.

Although this article concerns the *differential calculus*, its fundamental concepts can also be applied to the *integral calculus* (the work already done and its developments will be published in a future article).

*References:*

- [1] J. Bell, *A Primer of Infinitesimal Analysis*, Cambridge University Press, Cambridge, 1998.
- [2] J. Bell, *The Continuous and the Infinitesimal in mathematics and philosophy*, Polimetria, Monza, 2006.
- [3] A. Connes, Brisure de symétrie spontanée et géométrie du point de vue spectral, *Journal of Geometry and Physics*, **23** (1997), 206–234.
- [4] K. Hrbáček, Axiomatic foundations for nonstandard analysis, *Fund. Math.*, **98** (1978), 1–19.
- [5] K. Hrbáček, Nonstandard set theory, *Amer. Math. Monthly*, **86** (1979), 659–677.
- [6] P. Mancuso, *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century*, Oxford University Press, New York-Oxford, 1996.
- [7] I. Moerdijk and G. E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1991.
- [8] E. Nelson, Internal Set Theory: a new approach to Nonstandard Analysis, *Bulletin of the American Mathematical Society*, **83** (1977), 1165–1198.
- [9] E. Nelson, *Radically Elementary Probability Theory*, Annals of Mathematics Studies, PUP, Princeton, New Jersey, 1987.
- [10] A. Robinson, Non-standard Analysis, *Proc. of the Royal Academy of Sciences, Amsterdam*, ser. A, **64** (1961), 432–440.
- [11] A. Robinson, *Non-standard Analysis*, PUP, Princeton, New Jersey, rev.ed. 1996.
- [12] S. Sambursky, *Physics of the Stoics*, PUP, Princeton, New Jersey, first paperback ed. 1987.

Department of Mathematics, IST-University of Lisbon,  
 Av. Rovisco Pais, 1049-001 Lisboa, PORTUGAL

[jroquet@math.tecnico.ulisboa.pt](mailto:jroquet@math.tecnico.ulisboa.pt), [joseroquette@iol.pt](mailto:joseroquette@iol.pt)

Received: aa bb 20YY      Revised: cc dd 20ZZ